

# 2—D Quantum Oscillator

2025-04-02

This article explores the two-dimensional quantum harmonic oscillator through interactive visualizations. We consider a perturbation potential  $V(x, y) \propto xy$  and compute the first-order corrections to the energy levels due to the perturbation. We also show that the problem can be solved exactly by transforming the coordinates to the new coordinates  $\tilde{x}_1, \tilde{x}_2$  and  $\tilde{p}_1, \tilde{p}_2$  which are decoupled. The interactive plot allows readers to adjust quantum numbers  $n_x$  and  $n_y$  to visualize both the wavefunctions and probability densities in real-time.

blog: [https://tetraquark.vercel.app/posts/2d\\_quantum\\_harmonic\\_osc/](https://tetraquark.vercel.app/posts/2d_quantum_harmonic_osc/)

email: [quarktetra@gmail.com](mailto:quarktetra@gmail.com)

Consider a two-dimensional harmonic oscillator with the potential

$$V(x, y) = \frac{1}{2}m\omega^2(x^2 + y^2). \quad (1)$$

We want to consider a degenerate energy state with the eigenvalue  $3\hbar\omega$ , and compute the first-order corrections to the above energy level due to the perturbation  $H_1 = Kxy$ .

The level with energy  $3\hbar\omega$  is triple degenerate with the states  $|20\rangle$ ,  $|11\rangle$  and  $|02\rangle$ . We need to find the matrix elements of  $H_1 = kxy$  in this degenerate subspace basis. The matrix we have will be  $3 \times 3$  with 9 entries. But not all of them are nonzero (since  $x$  and  $y$  can change the state by 1 unit only), furthermore as  $xy$  is an Hermitian and real operator, the matrix will be symmetric (with respect to usual diagonal line). And finally the symmetry between  $x$  and  $y$  will force the matrix to be symmetric with respect to the other diagonal line. To sum up we need to calculate only 1 entry, others will be dictated by the symmetry, and will be the same. Let's calculate

$$\langle 11|kxy|20\rangle = k\langle 1|x|0\rangle\langle 1|y|2\rangle = \frac{k}{\sqrt{2}m\omega}. \quad (2)$$

The matrix we have is,

$$H_1 = \frac{k}{m\omega} \begin{pmatrix} 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \end{pmatrix} = \frac{k}{m\omega} J_x. \quad (3)$$

The eigenvalues are easy to get since  $J_x$  is in the spin-1 space, so it will have eigenvalues 0, 1, -1. For our matrix the eigenvalues are  $0, \pm \frac{k}{m\omega}$ , which are the first order corrections to the energy,  $3\omega$ . Now if we want to find the states corresponding to these energy levels we just find the eigenvectors of the above matrix, which are also the eigenvectors of  $J_x$ . Using the eigenvectors, we have the following first order corrected eigenstates of  $H$ ,

$$|0\rangle = \frac{|20\rangle - |02\rangle}{\sqrt{2}}, \quad |- \rangle = \frac{|20\rangle - \sqrt{2}|11\rangle + |02\rangle}{2}, \quad |+\rangle = \frac{|20\rangle + \sqrt{2}|11\rangle + |02\rangle}{2} \quad (4)$$

with the energies  $3\omega$ ,  $3\omega - \frac{k}{m\omega}$  and  $3\omega + \frac{k}{m\omega}$ , respectively. It is important to note that this problem could have been solved exactly by diagonalizing the potential  $V = \frac{1}{2}m\omega^2(x^2 + y^2) + kxy$  which can be done transforming the coordinates  $x, y$  to the coordinates  $\tilde{x}, \tilde{y}$ . The transformation is found by diagonalizing the matrix,

$$V_{ij} = \frac{\partial^2}{\partial x^i \partial x^j} V = \begin{pmatrix} m\omega^2 & k \\ k & m\omega^2 \end{pmatrix}. \quad (5)$$

The eigenvalues are  $m\omega^2 + k$  and  $m\omega^2 - k$  with the eigenvectors  $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$  and  $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ . What this algebra tells us is that if we define new coordinates  $\tilde{x}_1 = \frac{x+y}{\sqrt{2}}$  and  $\tilde{x}_2 = \frac{x-y}{\sqrt{2}}$ , then

$$\begin{aligned} V(x, y) &= \frac{1}{2} V_{ij} x^i x^j = \frac{1}{2} m\omega^2 (x^2 + y^2) + kxy \\ &= \frac{1}{2} \tilde{V}_{ij} \tilde{x}^i \tilde{x}^j = \frac{1}{2} (m\omega^2 + k) \tilde{x}_1^2 + \frac{1}{2} (m\omega^2 - k) \tilde{x}_2^2. \end{aligned} \quad (6)$$

You can check that the momentum part is transformed simply by renaming  $p_i$  by  $\tilde{p}_i$ . So the final Hamiltonian we have, is simply the Hamiltonian for two decoupled harmonic oscillators with different frequencies. The the total energy of the system is given by  $E = \sqrt{\omega^2 + \frac{k}{m}} (\tilde{n}_1 + \frac{1}{2}) + \sqrt{\omega^2 - \frac{k}{m}} (\tilde{n}_2 + \frac{1}{2})$ . Now as this is the exact answer, it must be equal to the approximate answer found before, at the first order in  $k$ . To compare we need to set the values  $\tilde{n}_1$  and  $\tilde{n}_2$  to the values which produce  $E = 3\omega$  when  $k = 0$ . The results are  $|\tilde{2}\tilde{0}\rangle$ ,  $|\tilde{1}\tilde{1}\rangle$  and  $|\tilde{0}\tilde{2}\rangle$ , but one must be careful that the states are not identical to the previous ones as the new pairs are in the new coordinates (that is why they have  $\sim$ ). The exact energy levels can be expanded at the first order as

$$\begin{aligned} E_{\tilde{1}\tilde{1}} &= \left[ \sqrt{\omega^2 + \frac{k}{m}} \left(1 + \frac{1}{2}\right) + \sqrt{\omega^2 - \frac{k}{m}} \left(1 + \frac{1}{2}\right) \right] = 3\omega + O(k^2), \\ E_{\tilde{2}\tilde{0}} &= \left[ \sqrt{\omega^2 + \frac{k}{m}} \left(2 + \frac{1}{2}\right) + \sqrt{\omega^2 - \frac{k}{m}} \left(\frac{1}{2}\right) \right] = 3\omega + \frac{k}{m\omega} + O(k^2), \\ E_{\tilde{0}\tilde{2}} &= \left[ \sqrt{\omega^2 + \frac{k}{m}} \left(\frac{1}{2}\right) + \sqrt{\omega^2 - \frac{k}{m}} \left(2 + \frac{1}{2}\right) \right] = 3\omega - \frac{k}{m\omega} + O(k^2). \end{aligned} \quad (7)$$

We note that the agreement of the exact results for energy with the perturbation theory results at the first order.

Figure 1 shows the probability density  $|\psi(x, y)|^2$  for the selected quantum numbers.

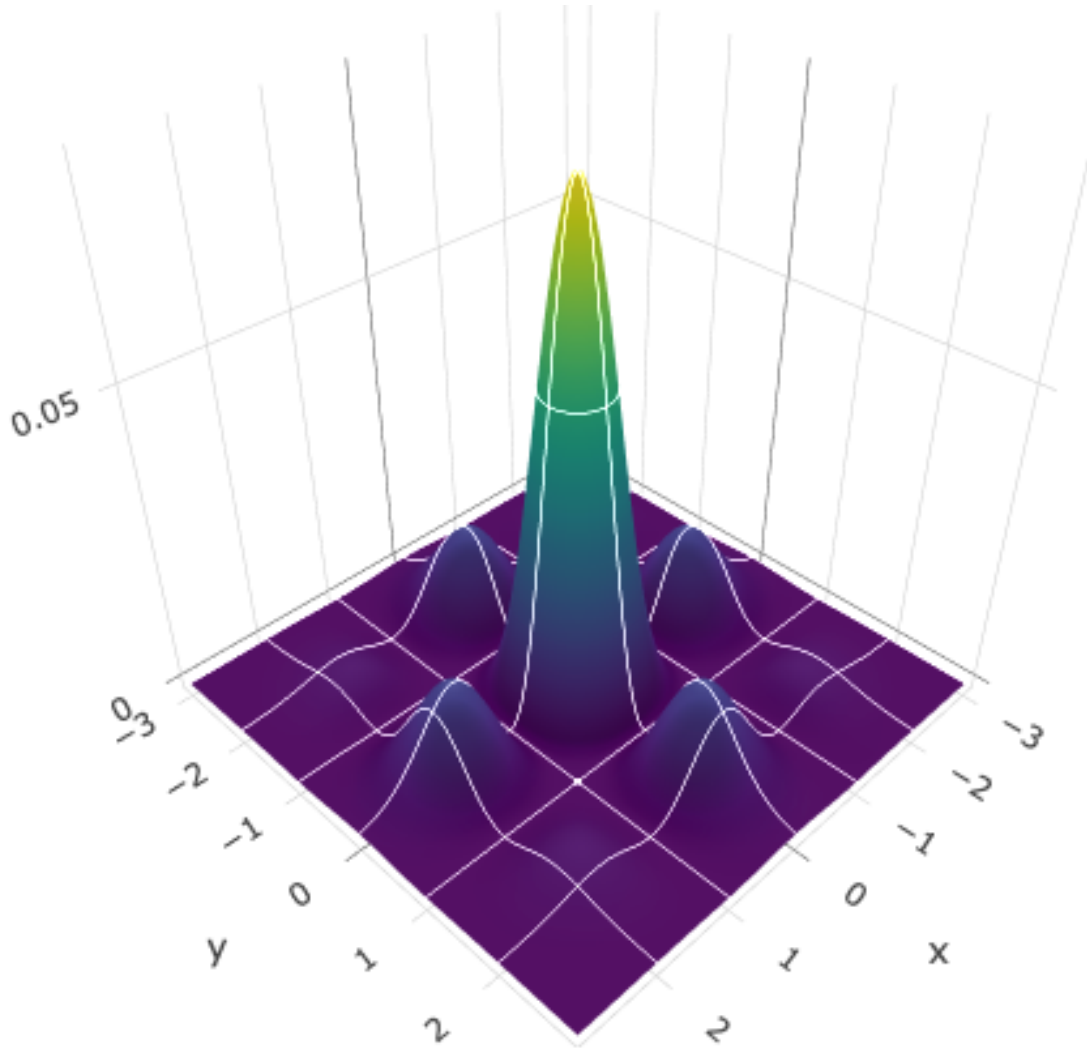


Figure 1: Wavefunction  $\psi(x, y)$  for  $n_x = 2 = n_y$ .