Simultaneous Eigenstates of angular momentum operators

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We show that if there is a simultaneous eigenstate of two components of the angular momentu, then this state has zero eigenvalues for all components.

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We are given that a state, say $|\psi\rangle$, is an eigenstate of the operators L_x and L_y, that is;

$$L_x|\psi\rangle = l_x|\psi\rangle, \quad L_y|\psi\rangle = l_y|\psi\rangle \tag{1}$$

We also know that;

$$[L_x, L_y] = i\hbar L_z. \tag{2}$$

Note that this is an operator relation which is valid on all kets. We can apply this operator equality onto our ket $|\psi\rangle$.

$$[L_x, L_y]|\psi\rangle = L_x L_y|\psi\rangle - L_y L_x|\psi\rangle = (l_x l_y - l_y l_x)|\psi\rangle = 0. \tag{3}$$

This implies, using Eq. 2

$$L_z|\psi\rangle = 0. \tag{4}$$

One can repeat the same steps;

$$[L_z, L_x]|\psi\rangle = L_z L_x|\psi\rangle - L_x L_z|\psi\rangle = 0 = i\hbar L_y|\psi\rangle. \tag{5}$$

The second method to solve the problem is to use uncertainty relations, which can be read from Shankar Chapter 9[1],

$$(\Delta\Omega)^2(\Delta\Lambda)^2 \geq \frac{1}{4}(\langle\psi|\{\hat{\Omega},\hat{\Lambda}\}|\psi\rangle)^2 + \frac{1}{4}|\langle\psi|[\Lambda,\Omega]|\psi\rangle|^2 \tag{6}$$

As the state we are considering is an eigenstate of operator L_x (L_y), associated uncertainty ΔL_x (ΔL_y) is zero. That means if we choose the arbitrary operators Λ and Ω as L_x and L_z, left hand side of Eq. 6 is zero. As both terms on right hand side are nonnegative, they both have to vanish, which implies that $\langle\psi|L_y|\psi\rangle = 0$. It is worth to emphasize that $\langle\psi|\Theta|\psi\rangle = 0$ does not imply $\Theta|\psi\rangle = 0$ for all Θ and $|\psi\rangle$. But in our case $\langle\psi|L_y|\psi\rangle = 0 = l_y\langle\psi|\psi\rangle$ which implies $l_y = 0$, since $\langle\psi|\psi\rangle \neq 0$. To prove that $l_x = 0$, we need to choose the arbitrary operators Λ and Ω as L_y and L_z, and follow the same steps.

[1] R. Shankar, *Principles of quantum mechanics*. New York, NY: Plenum, 1980 [Online]. Available: https://cds.cern.ch/record/102017