Dirac delta with a Function inside

2024-03-01

Dirac delta function appears frequently in physics. In certain cases, it takes a function as an argument. Such cases require care, and that is what we will take a quick look at in this post.

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This will be a quick post on handling functions inside Dirac-delta function. Let us start simple and figure out the constant scaling of the argument first, i.e., $\delta(\alpha\, x)$. This will certainly be proportional to $\delta(x)$ since the zero is still at $x = 0$. But it will pick up an overall factor since Dirac delta function is defined as a density, i.e., it integrates to 1. Let's see what $\delta(\alpha\, x)$ does under the integral:

$$\begin{aligned}\lim_{R\to\infty}\int_{-R}^{R} dx\delta(\alpha\, x) &= \lim_{R\to\infty}\int_{-R/\alpha}^{R/\alpha} d\left(\frac{y}{\alpha}\right)\delta(y) = \frac{1}{\alpha}\lim_{R\to\infty}\int_{-R/\alpha}^{R/\alpha} dy\delta(y)\\ &= \begin{cases}\frac{1}{\alpha}, & \text{if } \alpha > 0\\ -\frac{1}{\alpha}, & \text{if } \alpha < 0\end{cases} = \frac{1}{|\alpha|}, \end{aligned} \tag{1}$$

where the negative sign appears since we have to flip the integral limits when $\alpha < 0$.

Now consider a function $f(x)$ with zeros at points x_i. When we put this function inside $\delta()$, it will have peaks at $x = x_i$:

$$\delta\left(f(x)\right) = \sum_i c_i \delta(x - x_i), \tag{2}$$

and the goal is to find the c_i's. We expand $f(x)$ around x_i as $f(x) = (x - x_i)f'(x_i)$ when x is in the vicinity of x_i. This gives

$$\delta\left(f(x)\right)_{x\sim x_i} = \delta\left((x - x_i)f'(x_i)\right) = \frac{1}{|f'(x_i)|}\delta\left(x - x_i\right), \tag{3}$$

where we used Eq. 1. Putting it all together, we get:

$$\delta\left(f(x)\right) = \sum_i \frac{1}{|f'(x_i)|}\delta\left(x - x_i\right), \tag{4}$$

which expresses the Dirac-delta of a function as a series of basic Dirac-delta functions.