Musings on the Gamma Distribution

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In this post we demonstrate the derivation of the Gamma distribution through two different approaches. First, we show how the sum of n independent exponentially distributed random variables can be derived iteratively using convolution integrals. Starting with the case of $n = 2$, we explicitly calculate the probability density function and cumulative distribution function, then extend the result to $n = 3$ and generalize to arbitrary n. Second, we present an elegant alternative derivation using Laplace transforms, leveraging their properties to convert convolutions into multiplications in s-space. Both methods arrive at the same result, showing that the sum of n independent exponentially distributed random variables is Gamma distributed.

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$S_n = \sum_{i=1}^n T_i$ is a sum of n random numbers. It is illustrative to consider $n = 2$ case and figure out the distribution of the sum of two random numbers T_1 and T_2. The cumulative probability density of $S_2 \equiv T_1 + T_2$ is given by:

$$\begin{aligned} F_{S_2}(t) &= P(T_1 + T_2 < t) = \int_{t_1+t_2<t} f_{T_1}(t_1) f_{T_2}(t_2) dt_1 dt_2 = \int_{-\infty}^{\infty} \int_{-\infty}^{t-t_2} f_{T_2}(t_2) dt_2 f_{T_1}(t_1) dt_1 \\ &= \int_{-\infty}^{\infty} F_{T_2}(t - t_1) f_{T_1}(t_1) dt_1. \end{aligned} \tag{1}$$

The probability density function is the derivative of Eq. 1:

$$f_{S_2}(t) = \frac{d}{dt} F_{S_2}(t) = \int_{-\infty}^{\infty} f_{T_2}(t - t_1) f_{T_1}(t_1) dt_1 = \int_0^t f_{T_2}(t - t_1) f_{T_1}(t_1) dt_1, \tag{2}$$

where the limits of the integral are truncated to the range where $f \neq 0$. The integral in Eq.2 is known as the convolution integral:

$$f_{T_1} \circledast f_{T_2} \equiv \int_{-\infty}^{\infty} f_{T_2}(t - t_1) f_{T_1}(t_1) dt_1, \tag{3}$$

In the special case of exponential distributions, f is parameterized by a single parameter λ, which represents the failure rate, and it is given by

$$f_T(t) = \lambda e^{-\lambda t},\ t > 0. \tag{4}$$

From Eq. 2 we get:

$$f_{S_2}(t) = \int_0^t f_{T_2}(t-t_1) f_{T_1}(t_1) dt_1 = \lambda^2 \int_0^t e^{-\lambda(t-t_1)} e^{-\lambda t_1} dt_1 = \lambda^2 e^{-\lambda t} \int_0^t dt_1 = \lambda^2\, t\, e^{-\lambda t} \tag{5}$$

which is actually a Γ distribution. The corresponding cumulative failure function is:

$$\begin{aligned} F_{S_2}(t) &= \int_0^t d\tau f_{S_2}(\tau) = \lambda^2 \int_0^t d\tau\, \tau\, e^{-\lambda\tau} = -\lambda^2 \frac{d}{d\lambda}\left[\int_0^t d\tau\, e^{-\lambda\tau}\right] = \lambda^2 \frac{d}{d\lambda}\left[\frac{e^{-\lambda t}-1}{\lambda}\right] \\ &= 1 - e^{-\lambda t}(1+\lambda t). \end{aligned} \tag{6}$$

This is pretty neat. Can we move to the next level and add another T_i, i.e., $S_3 = T_1+T_2+T_3 = S_2+T_3$. We just reiterate Eq. 2 with probability density for S_2 from Eq. 5.

$$f_{S_3}(t) = \int_0^t f_{T_3}(t-t_1) f_{S_2}(t_1) = \lambda^3 \int_0^t e^{-\lambda(t-t_1)} t_1\, e^{-\lambda t_1} dt_1 = \lambda^3 \frac{t^2}{2} e^{-\lambda t}, \tag{7}$$

which was very easy! In fact, we can keep adding more terms. The exponentials kindly drop out of the t_1 integral, and we will be simply integrating powers of t_1, and for $S_n \equiv T_1+T_2+\cdots+T_n$ to get:

$$f_{S_n}(t) = \lambda^n \frac{t^{n-1}}{(n-1)!} e^{-\lambda t}. \tag{8}$$

It will be fun if we redo this with some advanced mathematical tools, such as the Laplace transform, which is defined as:

$$\tilde{f}(s) \equiv \mathcal{L}[f(t)] = \int_0^\infty dt\, e^{-s\,t} f(t). \tag{9}$$

There are a couple of nice features of the Laplace transforms we can make use of. The first one is the mapping of convolution integrals in t space to multiplication in s space. To show this, let's take the Laplace transform of Eq. 3:

$$\mathcal{L}[f_{T_1} \circledast f_{T_2}] = \int_0^\infty dt\, e^{-s\,t} \int_{-\infty}^\infty f_{T_2}(t-t_1) f_{T_1}(t_1) dt_1 = \int_{-\infty}^\infty dt_1 \int_0^\infty dt\, e^{-s\,(t-t_1)} f_{T_2}(t-t_1) e^{-s\,t_1} f_{T_1}(t_1). \tag{10}$$

Let's take a closer look at the middle integral:

$$\int_0^\infty dt\, e^{-s\,(t-t_1)} f_{T_2}(t-t_1) = \int_{-t_1}^\infty dt\, e^{-s\tau} f_{T_2}(\tau) = \int_0^\infty d\tau\, e^{-s\tau} f_{T_2}(\tau) = \tilde{f}_{T_2}(s), \tag{11}$$

where we first defined $\tau = t - t_1$, and then shifted the lower limit of the integral back to 0 since $f_{T_2}(t) = 0$ for $t < 0$. Putting this back in, we have the nice property:

$$\mathcal{L}[f_{T_1} \circledast f_{T_2}] = \tilde{f}_{T_1}(s)\tilde{f}_{T_2}(s). \tag{12}$$

How do we make use of this? The probability distribution of a sum of random numbers is the convolution of individual distributions:

$$f_{S_n} = \underbrace{f_{T_1} \circledast f_{T_2} \circledast \cdots \circledast f_{T_n}}_{n \text{ times}}. \tag{13}$$

We can map this convolution to multiplications in s space:

$$\tilde{f}_{S_n}(s) \equiv \mathcal{L}[f_{S_n}] = \underbrace{\tilde{f}_{T_1}\tilde{f}_{T_2}\cdots\tilde{f}_{T_n}}_{n \text{ times}} = \prod_{j=1}^{n} \tilde{f}_{T_j}. \tag{14}$$

When the individual random numbers are independent and have the same distribution, we get:

$$\tilde{f}_{S_n}(s) = \left(\tilde{f}_T\right)^n. \tag{15}$$

If the random numbers are exponentially distributed, as in Eq. 4, their Laplace transformation is easy to compute:

$$\tilde{f}(s) = \int_0^\infty dt\, e^{-s\,t} \lambda e^{-\lambda t} = \frac{\lambda}{s+\lambda}, \tag{16}$$

which means the Laplace transform of the sum is:

$$\tilde{f}_{S_n}(s) = \left(\frac{\lambda}{s+\lambda}\right)^n. \tag{17}$$

We will have to inverse transform Eq. 17, which will require some trick. This brings us to the second nifty property of Laplace transform. Consider transforming $tf(t)$:

$$\mathcal{L}[tf(t)] = \int_0^\infty dt\, te^{-s\,t} f(t) = -\frac{d}{ds}\left[\int_0^\infty dte^{-s\,t} f(t)\right] = -\frac{d}{ds}\left[\tilde{f}(s)\right]. \tag{18}$$

Therefore, we see that Laplace transform maps the operation of multiplying with t to taking negative derivatives in s space:

$$t \iff -\frac{d}{ds} \tag{19}$$

We re-write Eq. 17 as:

$$\tilde{f}_{S_n}(s) = \left(\frac{\lambda}{s+\lambda}\right)^n = \frac{\lambda^n}{(n-1)!}\left(-\frac{d}{ds}\right)^n\left(\frac{\lambda}{s+\lambda}\right). \tag{20}$$

Using the property in Eq. 19, we can invert the transform:

$$f_{S_n}(t) = \mathcal{L}^{-1}[f_{S_n}] = \quad = \quad \lambda^n \frac{t^{n-1}}{(n-1)!}e^{-\lambda t}, \tag{21}$$

which is what we got earlier in Eq. 8.