

Integral of the month: $\int dr \cos r^2$

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Fresnel integrals are a pair of integrals that are used to calculate the diffraction patterns of light waves. This post explores the mathematics of Fresnel integrals, a fundamental wave phenomenon that limits the resolution of optical instruments like telescopes. We first derive the results in the large r limit using the residue theorem. Then we will see how to evaluate the integrals numerically.

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The Fresnel integrals are defined as follows:

$$\begin{aligned} S(t) &= \int_0^t dr \sin r^2, \\ C(t) &= \int_0^t dr \cos r^2. \end{aligned} \tag{1}$$

For a general value of t , the integrals need to be evaluated numerically. However, the asymptotic values $C(t)$ and $S(t)$ can be calculated via the closed contour integral below:

$$I = \oint_C dz e^{-z^2}. \tag{2}$$

where the contour C is illustrated in Figure 1.

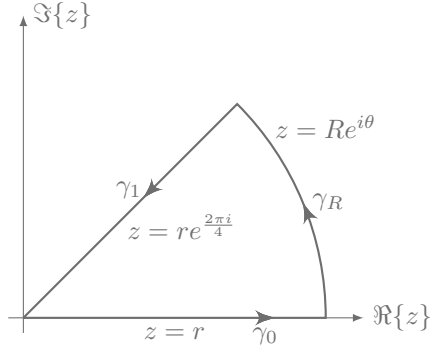


Figure 1: The contour to evaluate the integral. The return path, γ_1 , is chosen such that the integrand reduces to the regular Gaussian.

Let's first evaluate the integral on γ_0 in the limit $R \rightarrow \infty$:

$$I_{\gamma_0} = \lim_{R \rightarrow \infty} \int_0^R dr e^{-r^2} = \frac{\sqrt{\pi}}{2}, \quad (3)$$

where the details of the derivation can be found [here](#). Now consider the (absolute value of the) integral on γ_R in the limit $R \rightarrow \infty$:

$$\begin{aligned} |I_{\gamma_R}| &= \left| \lim_{R \rightarrow \infty} R \int_0^{\pi/4} d\theta e^{i\theta} e^{-R^2(\cos^2 \theta - \sin^2 \theta + 2i \cos \theta \sin \theta)} \right| = \left| \lim_{R \rightarrow \infty} R \int_0^{\pi/4} d\theta e^{i\theta + i \sin(2\theta)} e^{-R^2 \cos(2\theta)} \right| \\ &\leq \left| \lim_{R \rightarrow \infty} R \int_0^{\pi/4} d\theta e^{-R^2 \cos(2\theta)} \right|. \end{aligned} \quad (4)$$

Let's try to put a bound on $\cos(2\theta)$ in the range $0 \leq \theta \leq \pi/4$. At $\cos(2\theta)|_{\theta=0} = 1$ and $\cos(2\theta)|_{\theta=\pi/4} = 0$. We can draw a line that connects these two points: $1 - \frac{4\theta}{\pi}$. Since $\frac{d^2}{d\theta^2} \cos(2\theta) = -4\cos(2\theta) < 0$ for $0 < \theta < \pi/4$, we know that $\cos(2\theta) < 1 - \frac{4\theta}{\pi}$ in this range. This observation is illustrated in [?@fig-cosplot](#).

```

//| echo: false
//| label: fig-cosplot
//| fig-cap:  $\cos 2\theta$  and a bound on it with the line  $1 - \frac{4\theta}{\pi}$ 

cosPlot = {
  // Your existing plot div will be created and managed by Observable
  const element = DOM.element('div');
  element.style.width = "100%";
  element.id = "cosPlot";
  return element;
}

```

We can now go back to Eq. 6 and make use of the bound:

$$\begin{aligned}
|I_{\gamma_R}| &\leq \left| \lim_{R \rightarrow \infty} R \int_0^{\frac{\pi}{4}} d\theta e^{-R^2 \cos(2\theta)} \right| \leq \left| \lim_{R \rightarrow \infty} R \int_0^{\frac{\pi}{4}} d\theta e^{-R^2(1-\frac{4\theta}{\pi})} \right| \\
&= \left| \lim_{R \rightarrow \infty} R e^{-R^2} \int_0^{\frac{\pi}{4}} d\theta e^{R^2 \frac{4\theta}{\pi}} \right| \leq \left| \lim_{R \rightarrow \infty} R \frac{\pi}{4R^2} (1 - e^{-R^2}) \right| \\
&= \left| \lim_{R \rightarrow \infty} \frac{\pi}{4R} (1 - e^{-R^2}) \right| = 0.
\end{aligned} \tag{5}$$

Finally, let's look at the integral on γ_1 in the limit $R \rightarrow \infty$:

$$\begin{aligned}
I_{\gamma_1} &= \lim_{R \rightarrow \infty} R \int_R^0 dr e^{\frac{i\pi}{4}} e^{-r^2 \frac{i\pi}{2}} = -\frac{1+i}{\sqrt{2}} \lim_{R \rightarrow \infty} \int_0^R dr (\cos r^2 - i \sin r^2) \\
&= -\frac{1+i}{\sqrt{2}} \int_0^\infty dr (\cos r^2 - i \sin r^2) \\
&= -\frac{1}{\sqrt{2}} \left(\int_0^\infty dr \cos r^2 + \int_0^\infty dr \sin r^2 + i \left[\int_0^\infty dr \cos r^2 - \int_0^\infty dr \sin r^2 \right] \right). \tag{6}
\end{aligned}$$

As we have computed individual pieces of the integral Eq. 2, we can assemble them and state that they need to add to 0 since e^{-z^2} is analytic everywhere. Therefore we have:

$$\begin{aligned}
I &= \oint_C dz e^{-z^2} = 0 = I_{\gamma_0} + I_{\gamma_R} + I_{\gamma_1} \\
&= \frac{\sqrt{\pi}}{2} + 0 - \frac{1}{\sqrt{2}} \left(\int_0^\infty dr \cos r^2 + \int_0^\infty dr \sin r^2 + i \left[\int_0^\infty dr \cos r^2 - \int_0^\infty dr \sin r^2 \right] \right). \tag{7}
\end{aligned}$$

Matching the real and imaginary parts, we get:

$$\begin{aligned}
\frac{1}{\sqrt{2}} \left(\int_0^\infty dr \cos r^2 + \int_0^\infty dr \sin r^2 \right) &= \frac{\sqrt{\pi}}{2}, \\
\frac{1}{\sqrt{2}} \left(\int_0^\infty dr \cos r^2 - \int_0^\infty dr \sin r^2 \right) &= 0,
\end{aligned} \tag{8}$$

from which we get

$$\int_0^\infty dr \cos r^2 = \int_0^\infty dr \sin r^2 = \frac{1}{2} \sqrt{\frac{\pi}{2}} \simeq 0.626. \tag{9}$$

Now we can conclude with the plots of the Fresnel integrals in [?@fig-parametricplot](#) .

```
///| echo: false
///| label: fig-parametricplot
///| fig-cap: "Left: Fresnel integrals as a function of their argument, Right: parametric plot"

parametricPlot = {
  // Your existing plot div will be created and managed by Observable
  const element = DOM.element('div');
  element.style.width = "100%";
  element.id = "parametricPlot";
  return element;
}
```