Integral of the month: $\iint_S dt' dt\, f(t'-t)$

2025-04-13

When the function depends only on the difference of the parameters, we can simplify the double integral using a clever change of variables. This technique is essential for understanding the Wiener-Khinchin theorem and appears frequently in Fourier analysis.

blog: link

email: quarktetra@gmail.com

We want to compute the integral $I = \int_{\frac{-T}{2}}^{\frac{T}{2}} \int_{\frac{-T}{2}}^{\frac{T}{2}} dt' dt\, f(t'-t)$.

The argument of the function begs for a change of coordinates:

$$u = t' - t, \quad \text{and} \quad v = t + t', \tag{1}$$

and the associated inverse transform reads:

$$t' = \frac{u+v}{2}, \quad \text{and} \quad t = \frac{v-u}{2}. \tag{2}$$

This transformation will rotate and scale the integration domain as shown in Figure 1.

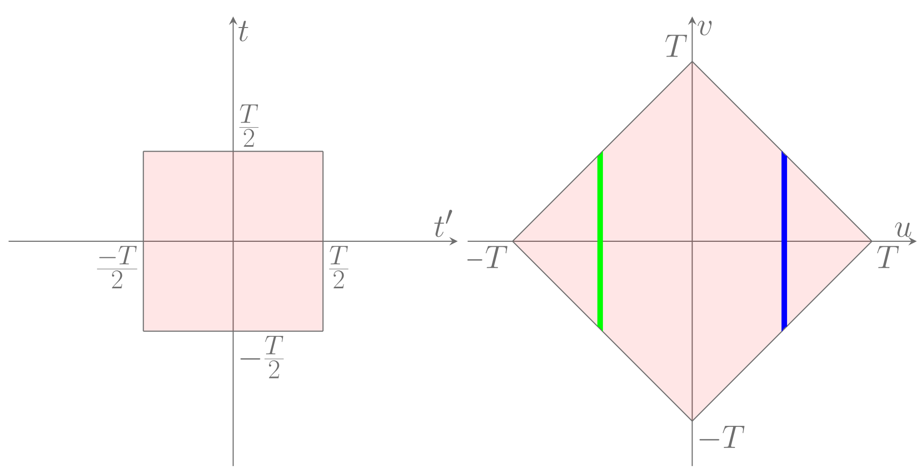

Figure 1: The integration domain in the $t-t'$ domain (left) and $u-v$ domain(right). Since there is no v dependence, v integration gives the height of the green and blue slices.

The equation of the top boundary on the right can be written as $v = T - u$, and on the left as \$ v= T+u\$. We can actually combine them as $v = T - |u|$. We can do the same analysis for the lower boundaries to see that the height of the slices at a given u is $2(T - |u|)$. This will help us easily integrate v out as follows:

$$\begin{aligned} I &= \int_{\frac{-T}{2}}^{\frac{T}{2}} \int_{\frac{-T}{2}}^{\frac{T}{2}} dt' dt f(t'-t) = \iint_{S_{u,v}} \left| \frac{\partial(t,t')}{\partial(u,v)} \right| dv du f(u) \\ &= \int_{-T}^{T} 2(T - |u|) \times \frac{1}{2} dv du f(u) = \int_{-T}^{T} du f(u)(T - |u|), \end{aligned} \tag{3}$$

where $\left| \frac{\partial(t,t')}{\partial(u,v)} \right| = \frac{1}{2}$ is the determinant of the Jacobian matrix associated with the transformation in Eq. 2.

In a typical problem, such as the proof of Wiener-Khinchin theorem, we need to evaluate the time average of the integral

$$\bar{I} = \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} du f(u)(T - |u|) = \int_{-\infty}^{\infty} du f(u), \tag{4}$$

where we assume that $uf(u)$ dies quickly enough so that the term drops out in the limit.