Interferometric detection of gravitational waves

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A gravitational wave changes the proper length of an interferometer arm. But, wouldn't it also change the wavelength of the travelling light? wouldnt that cancel the effect? If so, how can we use light to detect the wave?

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Introduction

A Michelson interferometer measures a gravitational wave by the difference in light-travel time between two perpendicular arms. The usual picture says the wave stretches one arm and compresses the other, so the light in one arm returns late and the light in the other returns early. The interference condition at the beam splitter moves.

That picture has a hole in it. If the distance stretches, the light that occupies the distance stretches too. A ruler that expands by the same fraction as the object it measures reports no change. For several decades this was not a pedantic objection. It was a reason to suspect that the whole effect was a choice of coordinates, with nothing at the photodetector.

Two separate claims got tangled together, and both were wrong for a precise reason.

The older claim is that gravitational waves themselves are a mathematical artifact. Eddington showed that some of the wave solutions written down in the 1910s propagate at an arbitrary coordinate speed, which he called the speed of thought [1]. Einstein and Rosen later convinced themselves that an exact wave metric was singular and therefore unphysical; the singularity was in the coordinates, and the corrected paper is the one that treats the waves as real [2]. What settles the question is not a cleverer gauge. It is the geodesic-deviation equation. The Riemann tensor of a wave accelerates neighboring free masses, and that relative acceleration can do irreversible work.

The newer claim is about the interferometer. Even after the waves were accepted as real, it remained unclear how light could serve as the ruler. Saulson put the question in the title of a short paper: if the light waves stretch, how can we use them to detect the gravitational wave [3]? The answer is that the instrument does not count how many wavelengths of the stored light fit into the arm. It floods the arm with new light, whose frequency was fixed by the laser, and it compares the arrival phase of that light with a local oscillator. Light already in the arm does stretch with the arm. Light emitted afterward does not, because the laser is small compared with the gravitational wavelength. The photodetector is continuously replacing the old field with the new one.

The rest of this note writes both arguments with the equations.

The wave equation

We start with the Einstein equation:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \mathcal{T}_{\mu\nu}, \tag{1}$$

and linearize it by writing the metric as a small departure from Minkowski space, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$ and $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. To first order in $h_{\mu\nu}$ the Ricci tensor is built from second derivatives of the perturbation. Impose the harmonic gauge on the trace-reversed field

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h, \qquad \partial^{\mu} \bar{h}_{\mu\nu} = 0, \tag{2}$$

where $h = \eta^{\mu\nu} h_{\mu\nu}$. In that gauge the linearized Ricci tensor collapses to

$$R_{\mu\nu} \simeq -\frac{1}{2}\, \partial_{\lambda} \partial^{\lambda} h_{\mu\nu}, \tag{3}$$

and (1) becomes the sourced wave equation of the stochastic-background calculation [4],

$$\partial_{\lambda} \partial^{\lambda} \bar{h}_{\mu\nu} = -16\pi G \mathcal{T}_{\mu\nu}. \tag{4}$$

An interferometer sits far from the binary, or from whatever else radiated. Locally $\mathcal{T}_{\mu\nu} = 0$, and (4) is the homogeneous equation

$$\Box \bar{h}_{\mu\nu} = 0. \tag{5}$$

A residual gauge transformation $x^{\mu} \to x^{\mu} + \xi^{\mu}$, with $\Box \xi^{\mu} = 0$, preserves the harmonic condition and can be used to impose

$$h_{0\mu} = 0, \qquad h^{\mu}{}_{\mu} = 0, \qquad \partial^{i} h_{ij} = 0. \tag{6}$$

What remains, for a wave traveling in the z direction, are two functions $h_{+}(t - z/c)$ and $h_{\times}(t - z/c)$. Everything else in the early catalogues of "solutions" could be removed by a choice of ξ^{μ}. Those pure-gauge pieces are what can be assigned any propagation speed one likes. They do not belong to the Riemann tensor.

Keep a plus-polarized wave and drop the propagation delay across the instrument, which is legitimate when the gravitational wavelength is long compared with the arms. The metric in the transverse-traceless gauge is

$$ds^{2} = -c^{2}dt^{2} + \bigl(1 + h(t)\bigr)\,dx^{2} + \bigl(1 - h(t)\bigr)\,dy^{2} + dz^{2}, \tag{7}$$

with $|h| \ll 1$. This is not a conformal rescaling. A single scale factor in front of $dx^{2}+dy^{2}+dz^{2}$ would stretch every direction together and would not be a vacuum wave. Here one axis stretches and the orthogonal axis compresses by the same fractional amount.

Geodesic deviation

The line element is the metric contracted with the coordinate differentials,

$$ds^{2} = g_{\mu\nu}\,dx^{\mu}dx^{\nu}. \tag{8}$$

Equation (7) is that statement with the components written out: $g_{tt} = -c^{2}$, $g_{xx} = 1 + h(t)$, $g_{yy} = 1 - h(t)$, and $g_{zz} = 1$. A length is a proper distance on a slice of constant time. Set $dt = 0$. Along the x arm, $dy = dz = 0$, so

$$ds = \sqrt{g_{xx}}\,dx = \sqrt{1 + h(t)}\,dx. \tag{9}$$

The proper distance between the beam splitter and the x mirror is the integral of that, not the coordinate location:

$$L_{x}(t) = \int_{0}^{L} ds = \int_{0}^{L} \sqrt{1 + h(t)}\,dx = L\left(1 + \frac{h(t)}{2}\right) + O(h^{2}). \tag{10}$$

The y arm gives $L_{y} = L(1 - h/2)$. The sum $L_{x} + L_{y}$ is unchanged through first order. The difference is the signal,

$$L_{x} - L_{y} = L\,h. \tag{11}$$

The same relative motion follows from the geodesic-deviation equation, which is written entirely in terms of the Riemann tensor and does not care which gauge produced it. For a separation ξ^x and four-velocity along ∂_t,

$$\frac{D^2\xi^x}{d\tau^2} = -R^x{}_{txt}\, c^2\, \xi^x. \tag{12}$$

For the metric (7) the relevant component is $R^x{}_{txt} = -\ddot{h}/(2c^2)$, so

$$\ddot{\xi}^x = \frac{1}{2}\ddot{h}\,\xi^x, \tag{13}$$

and likewise $\ddot{\xi}^y = -\frac{1}{2}\ddot{h}\,\xi^y$. Integrate twice, starting from rest at separation L when $h = 0$. The coordinate separation in the transverse-traceless gauge was constant, but the proper separation moves by exactly (10).

This is the content of the sticky-bead argument, given by Feynman at the 1957 Chapel Hill meeting and, independently, in the work of Bondi and collaborators [5], [6]. Thread two masses on a rod with friction, and align the rod with x. Equation (13) drives the masses apart and together. Friction opposes that relative motion and deposits heat in the rod. A coordinate transformation does not heat a rod. After 1957 the waves were no longer optional.

Light travel time

The interferometer does not read $L_x - L_y$ with a tape. It reads the time a light ray spends going out and back.

Set $ds = 0$ along the x arm, with $dy = dz = 0$:

$$c\,dt = \sqrt{1 + h(t)}\,dx = \left(1 + \frac{h(t)}{2}\right) dx + O(h^2). \tag{14}$$

Evaluate h on the unperturbed ray. That is the consistent first-order approximation: the correction to the path is itself proportional to h, and it multiplies another factor of h inside the integral. A ray that leaves the beam splitter at time t_1 reaches the far mirror at

$$t_m = t_1 + \frac{L}{c} + \frac{1}{2c}\int_0^L h\Big(t_1 + \frac{x}{c}\Big)\,dx, \tag{15}$$

and returns at

$$t_r = t_1 + \frac{2L}{c} + \delta t_x, \tag{16}$$

where the excess round-trip time, written at the moment t of return, is

$$\delta t_x(t) = \frac{1}{2c}\int_0^L \left[h\Big(t - \frac{x}{c}\Big) + h\Big(t - \frac{2L-x}{c}\Big)\right] dx. \tag{17}$$

The first term in the integrand is the return leg. The second is the outbound leg. The y arm has $1-h$ in the metric, so its excess is $-\delta t_x$ with the same formula. The differential delay is

$$\Delta t(t) \equiv \delta t_x(t) - \delta t_y(t) = 2\,\delta t_x(t). \tag{18}$$

If h varies slowly during a round trip, $h(t - x/c) \approx h(t)$ throughout the integral and

$$\delta t_x = \frac{L}{c}\,h, \qquad \Delta t = \frac{2L}{c}\,h. \tag{19}$$

The factor Lh/c is the one-arm excess. It equals $2(L_x - L)/c$, because the proper length shifted by $Lh/2$ and the light covers that extra length twice.

The three-term relation

Equation (17) is the special case in which the beam runs perpendicular to the wave. A beam at a general angle samples the wave three times, and the rate of change of its return time is fixed by those three samples [7].

Keep $c = 1$ for this section, and let θ be the angle between the beam and the propagation direction. The unperturbed ray out to the mirror is $x = s\sin\theta$, $z = s\cos\theta$, with s running from 0 to L. The plus polarization weights dx^2 against dy^2, so on this ray the null condition (14) becomes

$$dt = \left(1 + \frac{1}{2}h\sin^2\theta\right) ds. \tag{20}$$

The wave itself depends on $t - z$. On the way out, $z = s\cos\theta$ and $t = t_1 + s$, so the phase that enters h is $t_1 + s(1 - \cos\theta)$. On the way back the coordinate s runs from L down to 0, the elapsed time from the turning point is u, and the phase is $t_1 + L(1 - \cos\theta) + u(1 + \cos\theta)$. The excess round-trip time for a photon that leaves at t_1 is therefore

$$\delta t(t_1) = \frac{\sin^2\theta}{2}\int_0^L h(t_1 + s(1-\cos\theta))\,ds + \frac{\sin^2\theta}{2}\int_0^L h(t_1 + L(1-\cos\theta) + u(1+\cos\theta))\,du. \tag{21}$$

What a clock at the beam splitter records is not δt but how that delay changes from one photon to the next. Differentiate under the integral. Each integral of $\partial h/\partial t_1$ is a derivative with respect to its argument, so it collapses to the values of h at the endpoints. The Jacobian of the outbound argument is $1-\cos\theta$, and the Jacobian of the return argument is $1+\cos\theta$. Combined with $\sin^2\theta = (1-\cos\theta)(1+\cos\theta)$, the prefactors become

$$\frac{dt_r}{dt_1} = 1 + \frac{1}{2}\Big\{(1-\cos\theta)\,h(t_1+2L) - (1+\cos\theta)\,h(t_1) + 2\cos\theta\,h(t_1 + L(1-\cos\theta))\Big\}. \tag{22}$$

The three times are the emission, the reflection, and the return, read on the phase of the wave rather than on the laboratory clock. Equation (22) is Schutz's three-term relation. Restoring c replaces every interval L by L/c.

Two limits check it. If the beam is parallel to the wave, $\theta = 0$ or $\theta = \pi$, every coefficient sums to zero and $dt_r/dt_1 = 1$: a round trip spent running with the wave and then against it accumulates no net first-order delay. If the beam is perpendicular, $\theta = \pi/2$, only two of the three samples survive,

$$\frac{dt_r}{dt_1} = 1 + \frac{1}{2}[h(t_1+2L) - h(t_1)], \tag{23}$$

which is what one obtains by differentiating (17) with respect to the emission time. For a wave slow compared with L/c, $h(t_1 + 2L/c) - h(t_1) \approx (2L/c)\dot{h}$, and (23) says the excess itself is $(L/c)h$, in agreement with (19). A strictly constant h makes the right-hand side of (22) equal to 1. The static piece can be gauged away; only the change from one photon to the next is a signal.

An interferometer subtracts two copies of (22), one for each arm. For a wave incident along z on arms along x and y, both arms sit at $\theta = \pi/2$ and the plus polarization enters with opposite signs, which is why (18) is twice the one-arm delay rather than a difference of two unrelated angles. A source in a general direction assigns a different θ to each arm, and the detector tensor is that pair of three-term relations.

The phase at the beam splitter

The laser is a clock. Its angular frequency ω is a proper frequency, fixed by an atomic transition or by a reference cavity whose size is centimeters, not kilometers. Across that cavity the gravitational strain, integrated, is negligible: the gravitational wavelength at 100 Hz is several thousand kilometers. At the beam splitter, $dx = 0$, so (7) gives $d\tau = dt$. The coordinate frequency and the proper frequency are the same number.

A ray that returns at coordinate time t was emitted at $t - 2L/c - \delta t$. Relative to a local oscillator that keeps phase ωt, the returning field carries the phase

$$\delta\Phi_x(t) = \omega\, \delta t_x(t). \tag{24}$$

In the slow-wave limit, using $\omega/c = 2\pi/\lambda$,

$$\delta\Phi_x = \frac{2\pi L}{\lambda}\, h, \qquad \delta\Phi_x - \delta\Phi_y = \frac{4\pi L}{\lambda}\, h. \tag{25}$$

Equation (25) is the response of a plain Michelson, one round trip in each arm. LIGO's arms are $L = 4\,\mathrm{km}$, about 5×10^7 times the width of a human hair. With $\lambda = 1064\,\mathrm{nm}$ and a strain $h \sim 10^{-21}$, the arm difference is $\delta L = hL \approx 4 \times 10^{-18}\,\mathrm{m}$, or about 4×10^{-20} football fields, and the differential phase is of order $10^{-10}\,\mathrm{rad}$. The instruments that actually run use the arms as cavities, so the light makes that trip many times before it is allowed back out.

Arm cavities

I discusssed the details of the Fabry–Perot cavity in the Fabry–Perot post. Here I will just summarize the key results. The multiple-beam sum for a Fabry–Perot cavity is already worked out in the Fabry–Perot post: a wave of amplitude ε that crosses the input mirror, bounces, and returns picks up a round-trip factor $r_i r_e e^{i2\theta_T}$, and the geometric series sums to a circulating field

$$\varepsilon_{\mathrm{circ}} = \frac{t_i}{1 - r_i r_e e^{i2\theta_T}}\, \varepsilon. \tag{26}$$

At normal incidence the one-way phase defined there is $\theta_T = 2\pi L/\lambda$. A gravitational wave does not change the algebra. It changes L, and therefore θ_T. With the proper-length shift (10),

$$\delta\theta_T = \frac{2\pi}{\lambda}\frac{hL}{2} = \frac{\pi L}{\lambda}\, h. \tag{27}$$

The field that comes back toward the beam splitter is the reflected field of that post. For a lossless input mirror, $t_i^2 = 1 - r_i^2$, it is

$$\frac{\varepsilon_R}{\varepsilon} = \frac{r_i - r_e\, e^{i2\theta_T}}{1 - r_i r_e\, e^{i2\theta_T}}. \tag{28}$$

LIGO's mirrors are 40 kg slabs of fused silica, so one African elephant balances about 150 of them. LIGO runs the end mirror very close to a perfect reflector and the input mirror as the leak. The coefficients quoted in the Fabry–Perot post are $t_i^2 = 0.03$ and $r_e^2 = 0.99997$, which give a finesse

$$\mathcal{F} = \frac{\pi\sqrt{r_e r_i}}{1 - r_e r_i} \approx 208 \tag{29}$$

on the 4 km arms. On resonance $e^{i2\theta_T} = 1$ and, for $r_e = 1$, (28) sits at $\varepsilon_R/\varepsilon = -1$: every photon that entered eventually comes back, because the end mirror does not absorb it. Move slightly off resonance, $2\theta_T = 2\pi m + \psi$ with $\psi \ll 1$. Then

$$\frac{\varepsilon_R}{\varepsilon} \approx -1\left(1 + i\,\frac{1 + r_i}{1 - r_i}\,\psi\right), \tag{30}$$

so the phase of the returned light, measured from π, is

$$\delta\phi_R = \frac{1 + r_i}{1 - r_i}\,\psi \approx \frac{2\mathcal{F}}{\pi}\,\psi. \tag{31}$$

The round-trip phase produced by the strain is $\psi = 2\delta\theta_T = 2\pi L h/\lambda$, which is exactly $\delta\Phi_x$ in (25). The cavity multiplies it by $2\mathcal{F}/\pi$:

$$\delta\phi_{R,x} = \frac{4\mathcal{F}L}{\lambda}\,h, \qquad \delta\phi_{R,x} - \delta\phi_{R,y} = \frac{8\mathcal{F}L}{\lambda}\,h. \tag{32}$$

For $\mathcal{F} \approx 208$ the factor $2\mathcal{F}/\pi$ is about 130. The same series fixes how long a photon stays. After one round trip the circulating amplitude is multiplied by $r_i r_e$, and the round trip takes $2L/c$, so the stored amplitude falls as $e^{-t/\tau}$ with

$$\tau = -\frac{2L}{c\ln(r_i r_e)} \approx \frac{2L}{c(1 - r_i r_e)} \approx 1.8\,\text{ms} \tag{33}$$

for these mirrors. A photon that is already in the arm when the strain changes keeps its stretched wavelength until it leaks. A photon that enters later is launched at the laser frequency and only then starts accumulating (31). The cavity is the bookkeeping for that replacement,

done as a geometric series instead of one ray at a time. It is not a sealed box. The input mirror is the port through which the old field leaves and the new field arrives.

The cancellation that does not happen

Count wavelengths instead of timing crests. After the stretch, the proper arm length is $L(1 + h/2)$. Suppose the light already in the arm stretches by the same fraction, so its wavelength becomes $\lambda(1 + h/2)$. The number of wavelengths in a round trip,

$$N = \frac{2L_x}{\lambda_x} = \frac{2L}{\lambda}, \tag{34}$$

does not change. The interference condition, read this way, is blind to h. This is the correct conclusion for a standing wave that was stored in the arm and then subjected to a uniform, permanent strain, with no fresh light coming in and no clock outside the arm. It is the wrong conclusion for the interferometer that is actually operated.

The two descriptions diverge as soon as the laser keeps running.

Consider a step, $h(t) = h_0\Theta(t)$, as a cartoon of a strain that turns on and then holds. It is not a free wave, since a free wave must vary as $t - z/c$ and average to zero, but every term in (17) is linear, so the step shows the bookkeeping cleanly. Substitute into (17). For a return time $t \leq 0$ the entire integrand vanishes and $\delta t_x = 0$. For $0 < t < 2L/c$ only part of the outbound and return legs lie in the region $t' > 0$, and δt_x grows continuously. For $t > 2L/c$ both legs lie entirely after the step and

$$\delta t_x = \frac{L}{c}\, h_0. \tag{35}$$

At the instant the strain appears, the light-travel time has not yet changed. The crests already in the pipe are spaced by the old wavelength, and the pipe and those crests stretch together. Saulson's statement of this limit is that the instantaneous response of the light in the arm is null, and that the full delay (19) builds up over one light-travel time [3]. Equation (17) is that statement.

Now look at a photon emitted after the step. The laser did not stretch by a fractional amount $h_0/2$. Its proper frequency is still ω, so the wavelength it launches is still $\lambda = 2\pi c/\omega$. That new light travels a proper arm $L(1 + h_0/2)$. Its round-trip phase, relative to the local oscillator, is shifted by (24) with the delay (35). Nothing has stretched its wavelength to match the new length, because its wavelength was assigned at emission, in a region the wave does not strain.

An interferometer floods the arms. The field at the beam splitter at time t is the field that was emitted at $t - t_{rt}(t)$, not a fossil of the standing wave that occupied the arm before the metric changed. As the old light walks out and leaves, it is replaced by light whose phase records $\omega\delta t$. Comparing the new photons with the old ones is the entire measurement. A cavity that one fills and seals would show (34) and then nothing, until its stored light leaked and was replaced. The arm cavities are built to do that leaking on the storage time (33): the end mirror holds the light, and the input mirror is the hole through which the laser keeps the comparison going.

There is a second, independent reason the naive cancellation fails even before one distinguishes old light from new. The metric (7) stretches x and compresses y. There is no common factor to pull out of both arms. A conformal perturbation, $h_{ij} = h(t)\delta_{ij}$, would multiply every arm by the same $1 + h/2$, and the differential delay (18) would vanish. That perturbation is not a solution of the vacuum wave equation in the transverse-traceless gauge. The physical wave is traceless. The differential arm length (11) is the tracelessness, read at the beam splitter.

What the photodetector sees

Put the two arms back together. The field returning from each arm, referred to the local oscillator, is

$$E_x \propto \exp[-i\omega(t - \delta t_x(t))], \qquad E_y \propto \exp[-i\omega(t - \delta t_y(t))]. \tag{36}$$

An exact dark fringe is the wrong operating point. There the two arms differ in phase by π, the linear term in the leaked power cancels, and what remains is proportional to $(\delta\Phi)^2$. At $h \sim 10^{-21}$ that quadratic leak is hopeless. The instrument is held a small, fixed offset away from dark, so the power at the antisymmetric port moves linearly with the differential phase,

$$\delta P \propto \delta\Phi_x - \delta\Phi_y = \frac{4\pi L}{\lambda}\, h \tag{37}$$

in the slow-wave limit, and $2\mathcal{F}/\pi$ times larger once each arm is a cavity, by (32). More generally

$$\delta\Phi_x(t) - \delta\Phi_y(t) = \frac{2\omega}{c}\int_0^L \left[h\Big(t - \frac{x}{c}\Big) + h\Big(t - \frac{2L - x}{c}\Big)\right] dx, \tag{38}$$

which is (18) multiplied by ω. For a wave of frequency $f_{\rm gw}$ the round-trip time $2L/c$ is a fraction of a period when $f_{\rm gw} \ll c/(2L)$. Then (38) collapses to (25). At higher frequency the integral is a weighted average of h over the time the photon spent in the arm, and the response falls. That roll-off is the finite-arm transfer function. It is also the mathematical form of the

remark that the light has to travel through the changed arm before the phase knows the arm has changed.

The wavelength of the in-flight light does change. In the transverse-traceless picture this is the same fact as the cosmological redshift in comoving coordinates: crests that are already on a grid of freely falling masses are carried apart when the spatial metric grows [3]. Keeping that effect and also subtracting a separate "stretch of the ruler" from (38) counts it twice. The null geodesic already includes it. The phase that arrives is ω times the time of flight, with ω held by the laser.

Remarks

The historical artifact and the interferometer paradox are one mistake at two scales. In both cases a coordinate statement, "the masses do not move" or "the number of wavelengths does not change," was taken for an invariant one. The invariant statement about the masses is (13). The invariant statement about the light is (38): a proper frequency, defined where the wave has no room to act, multiplied by a coordinate time of flight that the wave does change.

Nothing here used a property of the electromagnetic field beyond the null geodesic and a phase referenced to a local clock. The same delay is what a bar detector registers as a tidal force on a resonant mass, with (13) as the driving term. The interferometer is that tidal measurement with the spring removed and the light-travel time left in its place.

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