LIGO Laser modulation

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We present a comprehensive analysis of laser modulation techniques used in the Laser Interferometer Gravitational-Wave Observatory (LIGO) for cavity length sensing and control. We examine the Pound-Drever-Hall (PDH) technique, which employs radio-frequency (RF) phase modulation of laser light at frequencies ranging from 9 MHz to 118 MHz to create optical sidebands. These sidebands interact with Fabry-Perot cavities to generate error signals for cavity locking.

blog: LIGO Laser Modulation email: quarktetra@gmail.com

LIGO uses laser light to measure the optical cavity lengths and the small changes due to gravitational waves. Each cavity is locked using the Pound-Drever-Hall (PDH) technique[1]. Laser light is modulated with Radio-frequency (RF) sidebands using electro-optic modulators. The RF sideband frequencies, (9MHz, 24MHz, 45MHz, and 118MHz) which are chosen to be resonant in some cavities and anti-resonant in others, create beats at various ports[2].

The performance of the photodetection depends on the operating frequency band and therefore it is beneficial to shift the operation point to an optimal frequency band. This is achieved via the RF modulation of the laser light, which is referred to as the optical heterodyne detection. In the heterodyne detection, the detection noise is expected to be dominated by the shot noise.

Upon the RF modulation, the laser light intensity becomes superposition of fields with different frequencies which results in a photodiode current with the same superposition of frequencies. This current signal is pushed through a set of filters to eliminate or select certain frequency bands. The selected signals are sent to mixers for demodulation to extract the signal associated with the length of the cavities. The resulting signal is used as a feedback to control the mirror position for locking.

The goal of this work is to optimize the electronic filters to maximize the SNR. Although it is not required for the filter optimization work, it is illustrative to discuss how the sidebands are injected and how they interact with the optical cavities, and that is what we are going to do in this section.

Phase Modulation

We will define the laser light as a plane wave propagating along the z axis. It can be written in the phasor notation as follows:

$$E_{\text{in}} = E_0 e^{i\omega_0 t - ikz}, \tag{1}$$

where E_0 is the amplitude and $\omega_0 = 2\pi f \approx 2\pi \times 2.8 \cdot 10^{14}$ Hz is the laser frequency.

The laser light is fed through a crystal as shown in Figure 1. A sinusoidal voltage is applied across the crystal which slightly modifies the crystal structure creating a phase shift in laser proportional to the signal.

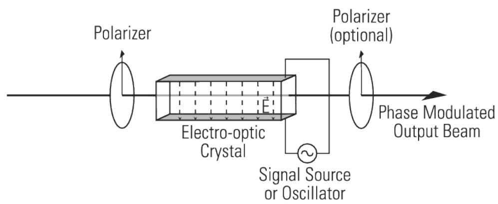

Figure 1: Block diagram for a phase modulation. Image Credit:newport.com

The resulting laser light picks up an additional phase at $z = 0$:

$$E = E_0 e^{i\omega_0 t + im\cos(\Omega t)} = E_0 e^{i\omega_0 t} \times e^{im\cos(\Omega t)}, \tag{2}$$

where m specifies the modulation strength and Ω is the angular frequency of the modulating signal with corresponding frequencies $f \sim$ 1MHz to 150MHz. It is important to note that the signal might pick up a relatively small, residual amplitude modulation in this process[3], but we will ignore that effect in this study. The modulated signal is shown in Figure 2 , where we take unrealistically large values of Ω relative to ω_o to make the effect visible in the plot.

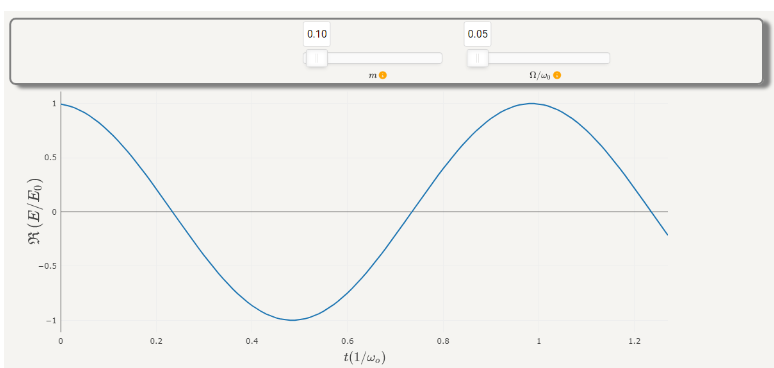

Figure 2: Modulated signal.

As this modulated laser light travels through the set up, it will also pick up additional phase due to the changes in the optical path length creating signal sidebands in the signal with frequencies in the range 1Hz to 10KHz. The RF sidebands are utilized for optical readout purposes, while the signal sidebands carry the signal to be measured like the gravitational-wave signal plus noise created in the interferometer.

We can expand out $\cos(\Omega t)$ in Eq. 2 using Euler's formula:

$$\cos(\Omega t) = \frac{e^{i\Omega t} + e^{-i\Omega t}}{2} \equiv \frac{z + 1/z}{2}, \tag{3}$$

where $z \equiv e^{i\Omega t}$. Rewriting Eq. 2 yields:

$$E = E_0 e^{i\omega_0 t} \times e^{im\frac{z+1/z}{2}} = E_0 e^{i\omega_0 t} \sum_{k=-\infty}^{\infty} i^k J_k(m) z^k = E_0 e^{i\omega_0 t} \sum_{k=-\infty}^{\infty} i^k J_k(m) e^{ik\Omega t}, \tag{4}$$

where we used the generating function for the Bessel functions of the first kind as illustrated in Figure 3

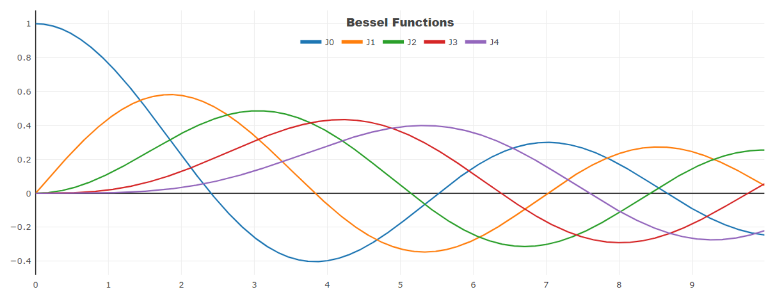

Figure 3: Bessel functions of the first kind.

The $k = 0$ term is the original field at frequency ω_0, i.e., it is the carrier. Additionally, we get infinitely many sidebands fields with frequencies shifted by $k\Omega$ relative to the carrier frequency. Although we have an infinite sum in Eq. 4, we can truncate it when $m \ll 1$ and keep the lowest order term on m, i.e., $k = \{0, \pm 1\}$ to get:

$$\begin{aligned} E &\simeq E_0 e^{i\omega_0 t} \left(J_0(m) - iJ_{-1}(m) e^{-i\Omega t} + iJ_1(m) e^{i\Omega t} \right) \\ &= E_0 e^{i\omega_0 t} \left(J_0(m) + iJ_1(m) e^{-i\Omega t} + iJ_1(m) e^{i\Omega t} \right), \end{aligned} \tag{5}$$

where we used $J_{-k}(m) = (-1)^k J_k(m)$. If we want, we can use the Taylor expansion for Bessel's functions [4]:

$$J_k(m) = \left(\frac{m}{2}\right)^k \sum_{n=0}^{\infty} \frac{\left(-\frac{m^2}{4}\right)^n}{n!(k+n)!} = \frac{1}{k!} \left(\frac{m}{2}\right)^k + O\left(m^{k+2}\right), \tag{6}$$

and insert this back into Eq. 5 to get

$$E = E_0 e^{i\omega_0 t}\left(1-\frac{m^2}{4}+i\frac{m}{2}\left(e^{-i\Omega t}+e^{i\Omega t}\right)\right) = E_0 e^{i\omega_0 t}\left(1-\frac{m^2}{4}+im\cos(\Omega t)\right), \quad (7)$$

If a more precise analysis required, more terms can be included resulting in higher harmonics of the modulating frequency relative to the carrier frequency. The laser light after modulation is sent to a Fabry-Perot interforemeter.

PDH Read out

A Fabry-Perot interferometer has a frequency dependent reflectivity coefficient, which is defined as the ratio of the reflected light amplitude and the incident light amplitude:

$$R(\omega) \equiv \frac{E_r}{E_i} = \frac{-r_1 + (r_1^2 + t_1^2) r_2 e^{i\phi}}{1 - r_1 r_2 e^{i\phi}}, \quad (8)$$

where r_1 and r_2 are the reflection coefficients, t_1 and t_2 are the transmission coefficients of the mirrors 1 and 2 of the cavity, and $\phi = 2\omega L/c$ is the phase picked up on the round trip.

A laser light with multiple frequency components, as in Eq. 5 can be used to probe the length of the Fabry-Perot interferometer, or it can used to generate a feedback signal to re-tune the frequency of the laser. Consider the set up in Figure 4, where a phase modulated signal is fed into the interferometer.

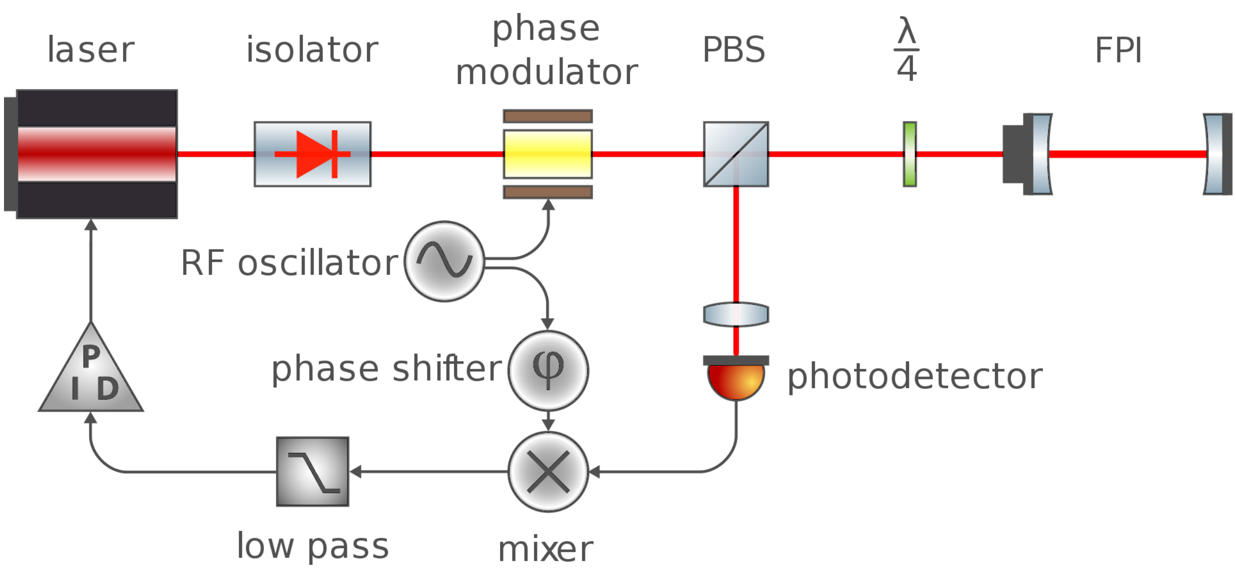

Figure 4: Pound–Drever–Hall(PDH) set up. Image Credit:Wikipedia

The incident light to the Fabry-Perot interferometer is given in Eq. 5, which we will rewrite as follows:

$$E_{\text{inc}} \simeq E_0 \left(J_0(m) e^{i\omega_0 t} + iJ_1(m) e^{i(\omega_0-\Omega)t} + iJ_1(m) e^{i(\omega_0+\Omega)t} \right). \tag{9}$$

The corresponding reflected light can be computed by multiplying each frequency term in Eq. 9 by the reflection coefficient at that frequency:

$$\begin{aligned} E_{\text{ref}} &\simeq E_0 \left(J_0(m) e^{i\omega_0 t} R(\omega_0) + iR(\omega_0-\Omega) J_1(m) e^{i(\omega_0-\Omega)t} + iR(\omega_0+\Omega) J_1(m) e^{i(\omega_0+\Omega)t} \right) \\ &= E_0 e^{i\omega_0 t} \left(J_0(m) R(\omega_0) + iR(\omega_0-\Omega) J_1(m) e^{-i\Omega t} + iR(\omega_0+\Omega) J_1(m) e^{i\Omega t} \right). \end{aligned} \tag{10}$$

The optical power associated with Eq. 10 is simply:

$$\begin{aligned} p_{\text{ref}} &\equiv E_{\text{ref}} E^*_{\text{ref}} \\ &= E_0^2 J_0^2(m) |R(\omega_0)|^2 + E_0^2 J_1^2(m) |R(\omega_0-\Omega)|^2 + E_0^2 J_1^2(m) |R(\omega_0+\Omega)|^2 \\ &\quad + 2E_0^2 J_0(m) J_1(m) \Re \left\{ R(\omega_0) R^*(\omega_0-\Omega) - R^*(\omega_0) R(\omega_0-\Omega) \right\} \sin(\Omega t) \\ &\quad + 2E_0^2 J_0(m) J_1(m) \Im \left\{ R(\omega_0) R^*(\omega_0-\Omega) - R^*(\omega_0) R(\omega_0-\Omega) \right\} \cos(\Omega t) \\ &\quad + (2\Omega\,\text{terms}). \end{aligned} \tag{11}$$

The photodiode will create a current proportional to p_{ref}, and it is the signal we are going to work with. We can isolate individual terms in Eq. 11 by multiplying the signal with $\cos(\Omega t)$ or $\sin(\Omega t)$ and integrating the result over a sufficiently long time window. If we multiple by $\cos(\Omega t)$ and and integrate, All the terms except for the $\cos(\Omega t)$ term in Eq. 11 will drop out. We can understand the signal we isolated better if we look at a case where the carrier signal is very close to the resonance frequency of the cavity. In this case, for a cavity of high quality the signals at frequencies $\omega_0 \pm \Omega$ will be totally reflected, i.e. $R(\omega_0 \pm \Omega) \simeq -1$. The reflected power will read:

$$p_{\text{ref}} = E_0^2 J_0^2(m) |R(\omega_0)|^2 + 2E_0^2 J_1^2(m) - 4E_0^2 J_0(m) J_1(m) \Im \left\{ R(\omega_0) \right\} \cos(\Omega t). \tag{12}$$

Furthermore, when we are very close to the resonance condition, we can drop the second order terms in $R(\omega)$ since they become very close to zero[5]. On the resonance, the phase change is very close to a multiple of 2π with a small deviation:

$$\phi = 2\pi N + 4\pi \frac{\delta L}{\lambda}. \tag{13}$$

where λ is the wavelength and δL is a small deviation. Inserting this back into Eq. 8 and expanding $R(\omega)$ in δL we get:

$$R(\omega) \equiv \frac{E_r}{E_i} = \frac{-r_1 + (r_1^2 + t_1^2) r_2 e^{i\phi}}{1 - r_1 r_2 e^{i\phi}} \simeq \frac{-r + r(1i + 4\pi\frac{\delta L}{\lambda})}{1 - r^2} = \frac{ir}{1 - r^2} 4\pi \frac{\delta L}{\lambda}, \tag{14}$$

which shows that the signal in the $\cos(\Omega t)$ channel is proportional to δL, which can be used as a feedback to either change the frequency of the laser or move the mirrors of the cavity to achieve locking. Sticking this back in Eq. 12, we get the final expression for the laser power:

$$p_{\text{ref}} = E_0^2 J_0^2(m) + 2E_0^2 J_1^2(m) - 4E_0^2 J_0(m) J_1(m) \frac{r}{1 - r^2} \frac{4\pi\delta L}{\lambda} \cos(\Omega t) + \sum_{k>1} \{k \times \Omega \text{ terms}\}, \tag{15}$$

where we added a set of terms to remind us that we will have all bunch of harmonics via the squaring operation and from the fact that this was an infinite sum of terms as in Eq. 4. If δL is what we are trying to measure, it will be the movement of the mirrors and the frequency content of the movement will be in the audio-band. This will be important in the SNR optimization process.

This completes the motivation for putting this section together: the information about the cavity length is embedded in the sidebands , and we want to be able to capture this information with notch filters and mixers while introducing minimal noise in the process.

Demodulation

The goal of this section is to see how the signal and noise move through the filters and mixers to the output. This will be critical as we dive deeper into the filter design. We will be mostly following the derivation from [6], and reproduce some of the results from that paper.

The laser light will be coupled to a photodiode, and assuming it is operating in the linear region, the photodiode current will be proportional to the laser power.

$$i(t) = \chi p(t) \equiv \sum_{n=-\infty}^{\infty} i_n(t) e^{i\Omega_n t}, \tag{16}$$

where we took the coefficients $i_n(t)$ time dependent because they are tied to δL, which is possibly time dependent, see Eq. 15. This will also make the math tractable since in the SNR calculations we will calculate power spectral density, and without the audioband frequency limited coefficients, we would run into difficulties of squaring Dirac-delta functions, and that is scary!

We will define the Fourier pairs with as usual and use the capital letters for the function in the frequency domain. The Fourier pair of $i_n(t)$ becomes:

$$I_n(\omega) = \mathcal{F}\{i_n(t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt e^{-i\omega t} i_n(t), \tag{17}$$

The Fourier transform of the full current is simply the sum of the shifted $I_n(\omega)$'s:

$$I(\omega) = \sum_{n=-\infty}^{\infty} \mathcal{F}\{i_n(t) e^{i\Omega_n t}\} = \sum_{n=-\infty}^{\infty} I_n(\omega - \Omega_n). \tag{18}$$

The photo-current will be pushed through a network of filters to reject/accept certain frequency component. The output will be a voltage, and we will denote it in the frequency domain as follows:

$$V(\omega) = Z(\omega) I(\omega), \tag{19}$$

where $Z(\omega)$ is the impedance of the complete filter circuit including internal impedance of the diode. This signal proceeds to a mixer, which multiplies the voltage in time domain with a demodulation signal $d(t)$:

$$v_{\text{mix}}(t) = d(t) v(t), \tag{20}$$

where $v(t)$ is the time domain voltage given by $\mathcal{F}^{-1}\{V(\omega)\}$. In the Fourier space, the multiplication becomes the convolution, hence we have:

$$V_{\text{mix}}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega' D(\omega - \omega') V(\omega'). \tag{21}$$

Sine wave demodulation

If the demodulation waveform is selected as a single tone we have:

$$d(t) = \sin(\Omega t - \gamma) \iff D(\omega) = \mathcal{F}\{d(t)\} = -i\sqrt{\pi/2}\left[e^{-i\gamma}\delta(\Omega - \omega) - e^{i\gamma}\delta(\Omega + \omega)\right]. \tag{22}$$

and this collapses the convolution integral:

$$V_{\text{mix}}(\omega) = -\frac{-ie^{i\gamma}}{2} V(\omega - \Omega) + \frac{ie^{i\gamma}}{2} V(\omega + \Omega). \tag{23}$$

The voltage at the output of the mixer is $V(\omega)$ shifted by the modulation frequency Ω. But we know from Eqs. 18 and 19 that the current and the voltage in the frequency domain are simply the sum of shifted $I_n(\omega)$'s, and therefore $V(\omega-\Omega) = I_{-1}(\omega)Z(\omega-\Omega)$, and $V(\omega+\Omega) = I_1(\omega)Z(\omega+\Omega)$, provided that the range of frequency content I_n is narrowband and the shifted elements have no overlap. Therefore, the output of the mixer in the frequency domain reads:

$$V_{\text{mix}}(\omega) = \frac{-ie^{-i\gamma}}{2} I_{-1}(\omega)Z(\omega-\Omega) + \frac{ie^{i\gamma}}{2} I_1(\omega)Z(\omega+\Omega). \tag{24}$$

Also Note that $\omega \ll \Omega$, which implies $Z(\omega-\Omega) \simeq Z(-\Omega) = Z^*(\Omega)$. Putting this back in:

$$V_{\text{mix}}(\omega) = \frac{-ie^{-i\gamma}}{2} I_{-1}(\omega)Z^*(\Omega) + \frac{ie^{i\gamma}}{2} I_1(\omega)Z(\Omega) = \frac{ie^{i\gamma}}{2} I_1(\omega)Z(\Omega) + \text{H.C.} \tag{25}$$

The corresponding time domain function is

$$v_{\text{mix}}(t) \simeq -\Im\left\{e^{i\gamma} i_1(t) Z(\Omega)\right\}. \tag{26}$$

We can simplify the final expression by defining Z as

$$Z(\Omega) = |Z(\Omega)|\, e^{i\beta}, \tag{27}$$

Furthermore, the initial phase of the modulation signal can be selected such that $i_1(t)$ is real, so that we get

$$v_{\text{mix}}(t) \simeq -|Z(\Omega)|\, i_1(t) \sin(\gamma-\beta), \tag{28}$$

which proves that the voltage at the mixer output is proportional to the audioband signal. The overall coefficient can be maximized by making $\gamma - \beta = (n+1/2)\pi$, n being an integer.

Square wave demodulation

In practice the demodulation is done with a square wave with a frequency that matches the original modulating signal. In fact, the demodulation signal can be an arbitrary function [6]. Consider a generic, periodic function $d(t)$ which can be expanded in Fourier series:

$$d(t) = \sum_{-\infty}^{\infty} d_n e^{in(\Omega t - \gamma)}. \tag{29}$$

For a square wave, for example, we have $d(t) = 1$ for $t < \pi/\Omega$, $d(t) = 0$ for $t \geq \pi/\Omega$, and repeated periodically. We can grab the d_m by using the orthogonality of the exponentials when integrated over a period:

$$\begin{aligned}
\int_0^{2\pi/\Omega} dt e^{-im(\Omega t)} &= \sum_{-\infty}^{\infty} d_n \int_0^{2\pi/\Omega} dt e^{in\Omega t} e^{-im\Omega t} \\
\int_0^{\pi/\Omega} dt 1 e^{-im(\Omega t)} &= \sum_{-\infty}^{\infty} d_n \frac{2\pi}{\Omega} \delta(n-m) = d_m \frac{2\pi}{\Omega} \\
\frac{1}{im\Omega}(e^{-im\pi} - 1) &= \sum_{-\infty}^{\infty} d_n \frac{2\pi}{\Omega} \delta(n-m) = d_m \frac{2\pi}{\Omega},
\end{aligned} \tag{30}$$

from which we get

$$d_m = \begin{cases} \frac{1}{i\pi m} & m \text{ is odd} \\ 0 & m \text{ is even} \end{cases}, \tag{31}$$

With this expansion, the convolution integral becomes a collection of terms:

$$V_{\text{mix}}(\omega) = \sum_{n=-\infty}^{\infty} d_n e^{in\gamma} Z(\omega - \Omega_n) I(\omega - \Omega_n) \simeq \sum_{n=-\infty}^{\infty} d_n e^{in\gamma} Z(\omega - \Omega_n) I_{-n}(\omega). \tag{32}$$

Injecting the noise

We have looked at how the signal propagates from its source all the way to output of the mixer. Let's now trace the path of the noise. We will consider two flavors of the noise following [6]: intensity noise and shot noise. In both cases, we injecting a variation in the laser power in Eq. 16 to get the corresponding variation in the current as:

$$\delta i(t) = \chi \delta p(t). \tag{33}$$

The total current becomes

$$j(t) = i(t) + \delta i(t), \tag{34}$$

where $\delta i(t)$ is the random noise in the current. We want to derive the power spectral density for the shot noise, and to get there we need to derive the autocorrelation function for δi and apply the Wiener-Khinchin Theorem. The auto correlation of the current noise is defined as:

$$C_i(t, t') = \mathbb{E}\left[\delta i(t), \delta i(t')\right], \tag{35}$$

and the finite period Fourier pair

$$C_i(\omega, \omega') \equiv \int_{-T/2}^{T/2} \frac{dt}{\sqrt{2\pi}} \int_{-T/2}^{T/2} \frac{dt'}{\sqrt{2\pi}} e^{-i\omega t} e^{i\omega' t'} C_i(t, t') \tag{36}$$

where T is the integration window. We will consider two cases for the shot noise.

Intensity noise

If the noise is stationary, i.e., $ C_i(t,t')=c_i(t-t')$, it results in the following Fourier pair:

$$\begin{aligned}
C_i(\omega,\omega') &= \int_{-T/2}^{T/2}\frac{dt'}{\sqrt{2\pi}}\int_{-T/2}^{T/2}\frac{dt}{\sqrt{2\pi}}e^{-i\omega t}e^{i\omega' t'}C_i(t,t') = \iint_D \frac{dudv}{2\pi}e^{-i(\omega+\omega')\frac{v}{2}-i(\omega-\omega')u}C_i(v)\\
&= \int_{-T}^{T} dv\left\{\frac{1}{2\pi}\int due^{-i(\omega-\omega')u}\right\}e^{-i(\omega+\omega')\frac{v}{2}}C_i(v)\\
&= \cdots \mathit{THEN\ A\ MIRACLE\ OCCURS}\cdots\\
&\simeq \int_{-T}^{T} dv\left\{\delta_T(\omega-\omega')\right\}e^{-i(\omega+\omega')\frac{v}{2}}C_i(v) = \sqrt{2\pi}\delta_T(\omega-\omega')\left\{\int_{-T}^{T}\frac{dv}{\sqrt{2\pi}}e^{-i\omega v}c_i(v)\right\}\\
&= \sqrt{2\pi}\delta_T(\omega-\omega')S_i(\omega). \qquad (37)
\end{aligned}$$

You might rightfully think that I *should be more explicit in the second step*: we defined new coordinates $u = t + t'$, and $v = t - t'$, recalculated the integration limits, and hid the ones for u integral since they will have some v dependence. We drop some oscillatory terms in large T limit.

Also note this equivalent representation:

$$\begin{aligned}
C_i(\omega,\omega') &\equiv \int_{-T/2}^{T/2}\frac{dt}{\sqrt{2\pi}}\int_{-T/2}^{T/2}\frac{dt'}{\sqrt{2\pi}}e^{-i\omega t}e^{i\omega' t'}C_i(t,t')\\
&= \mathbb{E}\left[\int_{-T/2}^{T/2}\frac{dt}{\sqrt{2\pi}}e^{-i\omega t}\delta i(t), \int_{-T/2}^{T/2}\frac{dt'}{\sqrt{2\pi}}e^{i\omega' t'}\delta i(t')\right]\\
&= \mathbb{E}\left[\delta I(\omega), \delta I^*(\omega')\right] = \sqrt{2\pi}\delta_T(\omega-\omega')S_i(\omega), \qquad (38)
\end{aligned}$$

which shows again that correlations will be not zero only if $\omega = \omega'$, and we will make use of this fact later.

The noise term in the current translates to the mixer output voltage just like the signal itself as in Eq. 24

$$\begin{aligned}
\delta V_{\text{mix}}(\omega) &= -\frac{-ie^{i\gamma}}{2}\delta V(\omega-\Omega) + \frac{ie^{i\gamma}}{2}\delta V(\omega+\Omega)\\
&= -\frac{ie^{-i\gamma}}{2}Z(\omega-\Omega)\delta I(\omega-\Omega) + \frac{ie^{i\gamma}}{2}Z(\omega+\Omega)\delta I(\omega+\Omega) \qquad (39)
\end{aligned}$$

The auto correlation function for noise in the voltage at the mixer output, $\delta v_{\text{mix}}(t)$, becomes:

$$c_{v_{\text{mix}}}(t,t') = \mathbb{E}\left[\delta v_{\text{mix}}(t), \delta v_{\text{mix}}(t')\right], \qquad (40)$$

and the corresponding Fourier pair reads:

$$\begin{aligned}
C_{v_{\rm mix}}(\omega,\omega') &= \mathbb{E}\left[\delta V_{\rm mix}(\omega), \delta V^*_{\rm mix}(\omega')\right] \\
&= \frac{1}{4}\left|Z(\omega-\Omega)\right|^2 \mathbb{E}\left[\delta I(\omega-\Omega), \delta I^*(\omega'-\Omega)\right] \\
&\quad +\frac{1}{4}\left|Z(\omega+\Omega)\right|^2 \mathbb{E}\left[\delta I(\omega'+\Omega), \delta I^*(\omega+\Omega)\right] \\
&= \sqrt{2\pi}\delta_T(\omega-\omega')\left\{\frac{1}{4}\left|Z(\omega-\Omega)\right|^2 S_i(\omega-\Omega)+\frac{1}{4}\left|Z(\omega+\Omega)\right|^2 S_i(\omega+\Omega)\right\} \\
&\equiv \sqrt{2\pi}\delta_T(\omega-\omega')S_{\rm mix}(\omega), \qquad (41)
\end{aligned}$$

where we defined

$$S_{\rm mix}(\omega) = \frac{1}{4}\left|Z(\omega-\Omega)\right|^2 S_i(\omega-\Omega)+\frac{1}{4}\left|Z(\omega+\Omega)\right|^2 S_i(\omega+\Omega). \qquad (42)$$

This shows the power of heterodyne detection: it shifts the intensity noise power frequency point to Ω from 0:

$$S_{\rm mix}(0) \simeq \frac{1}{2}\left|Z(\Omega)\right|^2 S_i(\Omega). \qquad (43)$$

Note that the final expression for the noised spectral density collapsed down to a single term because we considered single tone demodulation. For a generic demodulation signal we will have

$$S_{\rm mix}(\omega) = \sum_{n=-\infty}^{\infty} |d_n|^2 \left|Z(\omega-\Omega_n)\right|^2 S_i(\omega-\Omega_n). \qquad (44)$$

The take away from this section is that the heterodyne detection moves the noise point to a higher frequency at which the photodiode noise is expected to be much lower.

Shot noise

As we discussed earlier, the heterodyne detection moves the requency point at which the intensity noise is relatively low. At this the modulation frequency, the dominant noise is the shot noise. The shot noise is defined by the mean average value of the current, which is possibly time dependent since it carries the signal in the audioband. This results in a noise which is non-stationary. The correlation function is given by[6]:

$$C_i(\omega,\omega') = qI(\omega-\omega') \simeq 2\pi q \sum_{n=-\infty}^{\infty} I_n \delta_T(\omega-\omega'-\Omega_n) \qquad (45)$$

We need to recompute the voltage correlation function in Eq. 41 because we are now dealing with a non-stationary signal and there will be nonvanishing cross correlations in the current amplitudes. We will start from $V_{\rm mix}(\omega)$ as defined Eq. 24 and compute the autocorrelation

$$\begin{aligned}
C_{v_{\rm mix}}(\omega,\omega') &= \mathbb{E}\left[\delta V_{\rm mix}(\omega),\delta V^*_{\rm mix}(\omega')\right] \\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} d_n d_m^* e^{i(m-n)\gamma} Z(\omega-\Omega_n)Z^*(\omega'-\Omega_m)\mathbb{E}\left[I(\omega-\Omega_n),I^*(\omega-\Omega_n)\right] \\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} d_n d_m^* e^{i(m-n)\gamma} Z(\omega-\Omega_n)Z^*(\omega'-\Omega_m)C_i(\omega-\Omega_n,\omega'-\Omega_m) \\
&= 2\pi q \sum_{(n,m,k)=-\infty}^{\infty} d_n d_m^* e^{i(m-n)\gamma} Z(\omega-\Omega_n)Z^*(\omega'-\Omega_m)I_k \\
&\quad \times\delta_T(\omega-\omega'-\Omega_n+\Omega_m-\Omega_k),
\end{aligned} \tag{46}$$

which looks hopelessly too complicated, but we can eliminate one of the summations with the *delta* function because it will be non zero only if $-\Omega_n+\Omega_m-\Omega_k=0$ and $\omega=\omega'$ since $\omega \ll \Omega$. Putting this back in, we get:

$$C_{v_{\rm mix}}(\omega,\omega') = 2\pi q\delta(\omega-\omega')\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} d_n d_m^* e^{i(m-n)\gamma} Z(\omega-\Omega_n)Z^*(\omega'-\Omega_m)I_{m-n}. \tag{47}$$

This expression shows the frequencies at which the shot noise will appear.

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