Eigenvectors of $\hat{n} \cdot \vec{\sigma}$

2020-02-06

In this blog post, we explore an alternative method for finding the eigenvectors of the operator $\hat{n} \cdot \vec{\sigma}$, where $\vec{\sigma}$ represents the Pauli matrices and $\hat{n}$ is a unit vector. Rather than using conventional eigenvalue methods, we demonstrate how to obtain the eigenvectors through a series of rotations in spin space. This approach not only yields the correct results but also provides deeper insights into why the Pauli matrices transform as vector quantities under rotations.

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The straightforward way to find the eigenvectors of $\hat{n} \cdot \vec{\sigma}$ would be to use the usual method for finding eigenvalues and then the eigenvectors. Let us try to solve the problem using another method. We have $\hat{n} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$. Assume we start with $\hat{n}$ pointing along $\hat{z}$, so the state is $|\hat{z}_{up}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ which is an eigenvector of the $\vec{S} \cdot \hat{n}$ operator with eigenvalue 1. Let us rotate the state $|\hat{z}_{up}\rangle$ around $\hat{y}$ by angle θ which can be done by acting with the operator;

$$e^{-i\sigma_y\theta/2} = \begin{pmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix}. \tag{1}$$

You can check that above equation is correct by Taylor expanding the $e^{-i\sigma_y\theta/2}$, or you can visualize the effect as rotating a vector around $\hat{y}$ by angle θ keeping in mind that this is not really a vector (spin-1 particle), but it is a spinor (spin 1/2), which is reflected by the fact that we have $\frac{\theta}{2}$ instead of θ. Next task is to rotate again, around the $\hat{z}$ by angle ϕ which can be done by acting with the operator;

$$e^{-i\sigma_z\phi/2} = \begin{pmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{i\frac{\phi}{2}} \end{pmatrix}. \tag{2}$$

The composite operator becomes

$$
\begin{aligned}
e^{-i\sigma_z\phi/2}e^{-i\sigma_y\theta/2} &= \begin{pmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{i\frac{\phi}{2}} \end{pmatrix} \begin{pmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix} \\
&= \begin{pmatrix} e^{-i\frac{\phi}{2}}\cos(\frac{\theta}{2}) & -e^{-i\frac{\phi}{2}}\sin(\frac{\theta}{2}) \\ e^{i\frac{\phi}{2}}\sin(\frac{\theta}{2}) & e^{i\frac{\phi}{2}}\cos(\frac{\theta}{2}) \end{pmatrix}. \qquad (3)
\end{aligned}
$$

The eigenvectors can be recovered as

$$
\begin{aligned}
|\hat{n}+\rangle &= e^{-i\sigma_z\phi/2}e^{-i\sigma_y\theta/2}|\hat{z}_{up}\rangle = \begin{pmatrix} e^{-i\frac{\phi}{2}}\cos(\frac{\theta}{2}) \\ e^{i\frac{\phi}{2}}\sin(\frac{\theta}{2}) \end{pmatrix}, \\
|\hat{n}-\rangle &= e^{-i\sigma_z\phi/2}e^{-i\sigma_y\theta/2}|\hat{z}_{down}\rangle = \begin{pmatrix} -e^{-i\frac{\phi}{2}}\sin(\frac{\theta}{2}) \\ e^{i\frac{\phi}{2}}\cos(\frac{\theta}{2}) \end{pmatrix}. \qquad (4)
\end{aligned}
$$

In order to find $\langle\hat{n}\pm|\vec{S}|\hat{n}\pm\rangle$ we can use the above method to express $|\hat{n}\pm\rangle$ in terms of $|\hat{z}_{u,d}\rangle$.

$$
\langle\hat{n}\pm|\vec{S}|\hat{n}\pm\rangle = \langle\hat{z}_{u,d}|e^{i\sigma_y\theta/2}e^{i\sigma_z\phi/2}\vec{S}e^{-i\sigma_z\phi/2}e^{-i\sigma_y\theta/2}|\hat{z}_{u,d}\rangle. \qquad (5)
$$

To simplify the relation, we will compute the object $e^{i\sigma_j\alpha/2}\sigma_k e^{-i\sigma_j\alpha/2}$ where we will assume $k \neq j$ (if $k = j$, we can move σ_k through the exponentials to get σ_k). Consider $k \neq j$ case:

$$
\begin{aligned}
e^{i\sigma_j\alpha/2}\sigma_k e^{-i\sigma_j\alpha/2} &= \left(I\cos(\frac{\alpha}{2}) + i\sigma_j\sin(\frac{\alpha}{2})\right)\sigma_k\left(I\cos(\frac{\alpha}{2}) - i\sigma_k\sin(\frac{\alpha}{2})\right) \\
&= \cos\alpha\sigma_k - \sin\alpha\epsilon_{jkm}\sigma_m = \left(\cos\alpha\delta_{km} + \sin\alpha\epsilon_{kjm}\right)\sigma_m \\
&\equiv R^{(j)}_{km}(\alpha)\sigma_m. \qquad (6)
\end{aligned}
$$

This equation is nothing but the rotation equation for the vector $\vec{\sigma}$ around the j-axis. This tells us that $\vec{\sigma}$ indeed transforms like a vector, this is why it has a vector arrow on top! Now the problem becomes easier,

$$
\begin{aligned}
\langle\hat{n}\pm|S_k|\hat{n}\pm\rangle &= \langle\hat{z}_{u,d}|e^{i\sigma_y\theta/2}e^{i\sigma_z\phi/2}S_k e^{-i\sigma_z\phi/2}e^{-i\sigma_y\theta/2}|\hat{z}_{u,d}\rangle \\
&= \langle\hat{z}_{u,d}|e^{i\sigma_y\theta/2}R^{(z)}_{km}(\phi)S_m e^{-i\sigma_y\theta/2}|\hat{z}_{u,d}\rangle \\
&= R^{(z)}_{km}(\phi)R^{(y)}_{mn}(\theta)\langle\hat{z}_{u,d}|S_n|\hat{z}_{u,d}\rangle \\
&= \pm\frac{1}{2}R^{(z)}_{km}(\phi)R^{(y)}_{m3}(\theta). \qquad (7)
\end{aligned}
$$

We need to keep in mind that $R^{(j)}_{km}(\alpha) = \delta_{km}$ for $j = k$. Componentwise we get

$$
\begin{aligned}
\langle\hat{n}\pm|S_3|\hat{n}\pm\rangle &= \pm\frac{1}{2}R^{(z)}_{3m}(\phi)R^{(y)}_{m3}(\theta) = \pm\frac{1}{2}\delta_{3m}R^{(y)}_{m3}(\theta) = \pm\frac{1}{2}R^{(y)}_{33} = \pm\frac{1}{2}\cos\theta, \\
\langle\hat{n}\pm|S_2|\hat{n}\pm\rangle &= \pm\frac{1}{2}R^{(z)}_{2m}(\phi)R^{(y)}_{m3}(\theta) = \pm\frac{1}{2}\sin\theta\sin\phi, \\
\langle\hat{n}\pm|S_1|\hat{n}\pm\rangle &= \pm\frac{1}{2}R^{(z)}_{1m}(\phi)R^{(y)}_{m3}(\theta) = \pm\frac{1}{2}\sin\theta\cos\phi. \qquad (8)
\end{aligned}
$$

And these results can be combined into $\langle \hat{n} \pm | \vec{S} | \hat{n} \pm \rangle = \pm\frac{1}{2}\hat{n}$ As one can argue, this is not the fastest method to solve the problem, however it provides insights to σ- matrices and shows why they deserve the arrow on top. This comes from the fact that structure constants (ϵ_{ijk}) in the fundamental representation of $SU(2)$ group (the group of 2×2 matrices generated by σ-matrices), become the generators of the adjoint representation, i.e., the usual vector space.