Coherent states of Quantum Harmonic Oscillator

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A derivation of the coherent states of quantum harmonic oscillator.

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We start from the Schrödinger equation

$$H\psi(x,t) = i\hbar\frac{\partial}{\partial t}\psi(x,t). \tag{1}$$

The eigenstates of energy satisfy the following equation:

$$H\psi(x,t) = E\psi(x,t) = i\hbar\frac{\partial}{\partial t}\psi(x,t). \tag{2}$$

The differential equation is separable with the solution:

$$\psi(x,t) = e^{\frac{i}{\hbar}Et}\psi(x) \tag{3}$$

The classical Hamiltonian for particle of mass m and in a quadratic potential angular frequency ω reads

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 \tag{4}$$

where ω is the natural frequency of the oscillator. As we move from the classical system to the quantum system, we upgrade the position and momentum parameters to quantum operators:

$$x \to \hat{x}, \text{ and } p \to \hat{p} \tag{5}$$

where we added the "hat" to remind ourselves that these are operators. The quantum Hamiltonian becomes:

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2. \tag{6}$$

There are two main methods to calculate the energy eigenstates.

The hard way

We first follow the brute force method. We have a second order differential equation, and we bite the bullet and sit down to solve it. We can use the coordinate basis where $\hat{x}$ and $\hat{p}$ have the following representations:

$$\hat{x} = x, \text{ and } \hat{p} = -i\hbar \frac{d}{dx} \tag{7}$$

Therefore, the coordinate part of the Schrödinger equation becomes:

$$\left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m\omega^2 x^2 \right) \psi(x) = E\psi(x). \tag{8}$$

We can define a couple of dimensionless quantities $\chi = x\sqrt{\frac{m\omega}{\hbar}}$ and $\epsilon = \frac{2E}{\hbar\omega}$ to get:

$$\left(-\frac{d^2}{d\chi^2} + \chi^2 - \epsilon \right) \psi(x) = 0. \tag{9}$$

We should first try to understand the asymptotic solution where $\chi^2 \gg \epsilon$:

$$\left(-\frac{d^2}{d\chi^2} + \chi^2 \right) \psi(x) \simeq 0. \tag{10}$$

This equation has a special solution called parabolic cylinder functions. However, since we are looking for the asymptotic solutions, we can make an educated guess of the form $e^{-\alpha\chi^2}$ and plug it in to find that $\alpha = 1/2$ solves the differential equation at the first order. Or, we can try to split the $-\frac{d^2}{d\chi^2} + \chi^2$ operator into two first order operators and drop a small term in the large χ limit:

$$\left(-\frac{d^2}{d\chi^2} + \chi^2 \right) \psi(x) \simeq \left(-\frac{d}{d\chi} + \chi \right) \left(\frac{d}{d\chi} + \chi \right) \psi(x) \simeq 0. \tag{11}$$

We then combine the asymptotic solution with a yet-unknown function and propose a solution of the following form:

$$\psi(\chi) = e^{-\frac{\chi^2}{2}} h(\chi) \tag{12}$$

up to the normalization constant, which we will calculate later. Plugging this back into \ref{eq:seqpos2}, we get the following second order differential equation.

$$\left(-\frac{d^2}{d\chi^2} + 2\chi \frac{d}{d\chi} + 1 - \epsilon \right) h(\chi) = 0. \tag{13}$$

We can now try a power series of the form

$$h(\chi) = \sum_{j=0}^{\infty} c_j \chi^j \tag{14}$$

Inserting this back in, we get:

$$-\sum_{j=0}^{\infty} j(j-1)c_j\chi^{j-2} + \sum_{j=0}^{\infty}(2j+1-\epsilon)c_j\chi^j = 0 \tag{15}$$

Since the first two terms in the first summation vanish due to the $j(j-1)$ coefficient, we can start the first summation index, j, from 2, and redefine j as $j+2$ and pull the starting point back to 0:

$$-\sum_{j=0}^{\infty}(j+2)(j+1)c_{j+2}\chi^j+\sum_{j=0}^{\infty}(2j+1-\epsilon)c_j\chi^j = \sum_{j=0}^{\infty}\left(-(j+2)(j+1)c_{j+2} + (2j+1-\epsilon)c_j\right)\chi^j = 0 \tag{16}$$

In order to set this to zero we need to have the recurrence equation:

$$c_{j+2} = \frac{2j+1-\epsilon}{(j+2)(j+1)}c_j \tag{17}$$

Note that this is problematic because the coefficients are not decaying fast enough. In fact, this relation implies that $h(\chi) \propto e^{\chi^2}$, and even the prefactor $e^{-\chi^2/2}$ will not be decaying fast enough to make the wavefunction normalizable. The only way out of this is to truncate the series at some point. Remember that the only knob we have is ϵ, and we can set it an integer value such that when $2j+1=\epsilon$, the series terminates. This is a profound finding because the physicality of the solution requires the quantization of the energy. Going back to the original parameters, $E = \frac{\epsilon\hbar\omega}{2}$, we can write the energy as:

$$E = \hbar\omega(n+\frac{1}{2}) \tag{18}$$

There is another subtle problem: note that the recurrence formula relates c_0 to c_2, c_2 to c_4 and so forth, and c_1 to c_3, c_3 to c_5. In other words, the only free coefficients are c_0 and c_1. As we discussed earlier, we can truncate the series by selecting ϵ appropriately. However, we have only one degree of freedom in ϵ, and we cannot use that to terminate both of the odd and even series at the same time. That means only one of the coefficients c_0 and c_1 can be nonzero at the same time. This is also expected from the parity symmetry of the Hamiltonian: it stays invariant under $x \to -x$, which implies that solutions should stay invariant up to the sign. Therefore, odd and even powers of χ cannot mix in the energy eigenstates.

Since the series in Eq. 14 will terminate at $j = n/2$ due to the recurrence relation in \ref{eq:recurr}, it is convenient to redefine the summation index $s = \frac{n}{2} - j$, and rewrite the solution as a sum of a finite number of terms:

$$h_n(\chi) = \sum_{s=0}^{\lfloor n/2 \rfloor}(-1)^s\frac{n!}{(n-2s)!s!}(2x)^{n-2s}, \tag{19}$$

where $(-1)^s$ originates from flipping the sign of the numerator in 17, and powers of 2 originate from $n/2$'s in the denominator. These are Hermite polynomials, which can be written explicitly as

$$
\begin{aligned}
h_0(\chi) &= 1 \\
h_1(\chi) &= 2x \\
h_2(\chi) &= 4x^2 - 2 \\
h_3(\chi) &= 8x^3 - 12x \\
h_4(\chi) &= 16x^4 - 48x^2 + 12 \qquad (20)\\
&\vdots \qquad (21)
\end{aligned}
$$

Now we have to deal with the normalization of the wavefunction in 12. There is a very elegant way of doing this via the generating function. Let's multiply \ref{eq:hermite) with $\frac{t^n}{n!}$ and sum over n:

$$g(\chi,t) \equiv \sum_{n=0}^{\infty} \frac{t^n}{n!} h_n(\chi) = \sum_{n=0}^{\infty} \sum_{s=0}^{\lfloor n/2 \rfloor} (-1)^s \frac{t^n}{(n-2s)!s!} (2\chi)^{n-2s} = \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} (-1)^s \frac{t^n}{(n-2s)!s!} (2\chi)^{n-2s} \tag{22}$$

where we extended the summation upper limit since we will negative factorials give negative infinities killing all the terms for $s > n/2$. Now define $n - 2s = m$ and do some shuffling:

$$g(\chi,t) = \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} (-1)^s \frac{t^{m+2s}}{m!s!} (2\chi)^m = \sum_{s=0}^{\infty} \frac{(-t^2)^s}{s!} \sum_{m=0}^{\infty} \frac{(2t\chi)^m}{m!} = e^{-t^2+2t\chi}. \tag{23}$$

Now it becomes an easy task to compute the normalization factor. Consider the following:

$$\int_{-\infty}^{\infty} d\chi e^{-\chi^2} g(\chi,t) g(\chi,q) = \int_{-\infty}^{\infty} d\chi e^{-\chi^2-t^2+2t\chi-q^2+2q\chi} = \int_{-\infty}^{\infty} d\chi e^{-(\chi-t-q)^2+2qt} = \sqrt{\pi} e^{2qt} = \sqrt{\pi} \sum_{n=0}^{\infty} \frac{(2qt)^n}{n!} \tag{24}$$

and evaluate the integral in the series expansion:

$$
\begin{aligned}
\int_{-\infty}^{\infty} d\chi e^{-\chi^2} g(\chi,t) g(\chi,q) &= \int_{-\infty}^{\infty} d\chi e^{-\chi^2} \sum_{n=0}^{\infty} \frac{t^n}{n!} h_n(\chi) \sum_{m=0}^{\infty} \frac{q^m}{m!} h_m(\chi) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{t^n}{n!} \frac{q^m}{m!} \int_{-\infty}^{\infty} d\chi e^{-\chi^2} h_n(\chi) h_m(\chi) \\
&= \sum_{n=0}^{\infty} \frac{(qt)^n}{(n!)^2} \int_{-\infty}^{\infty} d\chi e^{-\chi^2} h_n(\chi) h_n(\chi) + \sum_{n=0}^{\infty} \sum_{m=0, m \neq n}^{\infty} \frac{t^n}{n!} \frac{q^m}{m!} \int_{-\infty}^{\infty} d\chi e^{-\chi^2} h_n(\chi) h_m(\chi) \qquad (25)
\end{aligned}
$$

By comparing the coefficients of qt terms in Eqs. 24 and 25, we first see that the cross terms should vanish. We also get the normalization constant:

$$\int_{-\infty}^{\infty} d\chi e^{-\chi^2} h_n(\chi) h_m(\chi) = 2^n \sqrt{\pi} n! \delta_{n,m}. \tag{26}$$

The orthogonality of the eigenfunctions is not a coincidence since the differential operator we are dealing with can be transformed into a self-adjoint form, and the orthogonality is guaranteed due to the Sturm-Liouville theory[1]. Putting it all together we have the normalized energy eigenstates:

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} h_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) e^{-\frac{m\omega x^2}{2\hbar}}, \quad n = 0, 1, 2, ... \quad . \tag{27}$$

It is operationally more practical to combine $\hat{x}$ and $\hat{p}$ operators into raising and lowering ladder operators. The harder method is based on the recurrence relations of the Hermite polynomials. Taking the derivative of the equality in 22 with respect to χ, we get:

$$\begin{aligned} \frac{\partial}{\partial\chi} g(\chi, t) &= 2te^{-t^2+2t\chi} = 2\sum_{n=0}^{\infty} \frac{t^{n+1}}{n!} h_n(\chi) = 2\sum_{m=1}^{\infty} \frac{t^m}{(m-1)!} h_{m-1}(\chi) = 2\sum_{m=1}^{\infty} \frac{mt^m}{m!} h_{m-1}(\chi) = 2\sum_{n=0}^{\infty} \frac{nt^n}{n!} h_{n-1}(\chi) \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} h'_n(\chi), \end{aligned} \tag{28}$$

where we first defined $m = n + 1$, and then relabeled m as n. We also added the vanishing $n = 0$ term in the summation to make the sum start from 0. Matching the coefficients of t^n terms, we get the first recurrence relation of the Hermite functions:

$$2nh_{n-1}(\chi) = h'_n(\chi). \tag{29}$$

Let's try taking the derivative with respect to t to get:

$$\begin{aligned} \frac{\partial}{\partial t} g(\chi, t) &= (-2t + 2\chi)e^{-t^2+2t\chi} = -2\sum_{n=0}^{\infty} \frac{t^{n+1}}{n!} h_n(\chi) + 2\sum_{n=0}^{\infty} \frac{t^n}{n!} \chi h_n(\chi) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \{2\chi h_n(\chi) - 2nh_{n-1}(\chi)\} \\ &= \sum_{n=0}^{\infty} n\frac{t^{n-1}}{n!} h_n(\chi) = \sum_{n=0}^{\infty} \frac{t^n}{n!} h_{n+1}(\chi). \end{aligned} \tag{30}$$

Matching the coefficients of t^n terms, we get the second recurrence relation of the Hermite functions:

$$h_{n+1}(\chi) = 2\chi h_n(\chi) - 2nh_{n-1}(\chi). \tag{31}$$

We can combine Eqs. 29 and 31 to get another flavor:

$$h_{n+1}(\chi) = \left(2\chi - \frac{d}{d\chi}\right) h_n(\chi). \tag{32}$$

Now consider the following operator acting on $\psi_n(\chi)$ as it is defined Eq. 27:

$$\frac{1}{\sqrt{2}}\left(\chi-\frac{d}{d\chi}\right)\psi_n(\chi) = \frac{1}{\sqrt{2}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\frac{1}{\sqrt{2^n n!}}\left(\chi-\frac{d}{d\chi}\right)\left(h_n(\chi)e^{-\frac{\chi^2}{2}}\right)$$
$$= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\frac{\sqrt{n+1}}{\sqrt{2^{n+1}(n+1)!}}e^{-\frac{\chi^2}{2}}\left(2\chi-\frac{d}{d\chi}\right)h_n(\chi) = \sqrt{n+1}\psi_{n+1}(\chi) \quad (33)$$

where we used Eqs. 32.

Now consider another operator acting on $\psi_n(\chi)$:

$$\frac{1}{\sqrt{2}}\left(\chi+\frac{d}{d\chi}\right)\psi_n(\chi) = \frac{1}{\sqrt{2}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\frac{1}{\sqrt{2^n n!}}\left(\chi+\frac{d}{d\chi}\right)\left(h_n(\chi)e^{-\frac{\chi^2}{2}}\right) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\frac{1}{\sqrt{2^{n+1}n!}}e^{-\frac{\chi^2}{2}}h_n'(\chi)$$
$$= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\frac{1}{2\sqrt{n}\sqrt{2^{n-1}(n-1)!}}e^{-\frac{\chi^2}{2}}2nh_{n-1}(\chi) = \sqrt{n}\psi_{n-1}(\chi), \quad (34)$$

where we used Eqs. 29. The operators $\frac{1}{\sqrt{2}}\left(\chi\pm\frac{d}{d\chi}\right)$ can be written in terms of x and $\hat{p}$ and they will be called a and $a^\dagger$, and that would be how one solves the harmonic oscillator the hard way. Now let's look into the method of operators.

The operational way

We can define the ladder operators as follows:

$$a = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega x+ip), \quad a^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega x-ip) \iff \hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger), \quad \hat{p} = -i\sqrt{\frac{m\hbar\omega}{2}}(a-a^\dagger). \quad (35)$$

The commutation relation $[x,p]=i\hbar$ turns to

$$[a,a^\dagger] = 1. \quad (36)$$

The Hamiltonian becomes

$$H \equiv \hbar\omega(a^\dagger a+\frac{1}{2}). \quad (37)$$

Comparing Eq. 37 with Eq. 18 we can associate $a^\dagger a$ with number operator:

$$N = a^\dagger a, \quad (38)$$

which returns the state number:

$$N|n\rangle = n|n\rangle. \quad (39)$$

Let's now figure out how a and $a^\dagger$ act on eigenstate $|n\rangle$. We can read the energy value by acting on the new state with H:

$$Ha|n\rangle = (aH+[H,a])|n\rangle = (aH-a\hbar\omega)|n\rangle = \hbar\omega\left(n-1+\frac{1}{2}\right)a|n\rangle, \quad (40)$$

which shows that the state $a|n\rangle$ can be indexed as $n-1$, i.e., $a|n\rangle = c|n-1\rangle$ where c is the normalization constant. The overall coefficient c can be calculated as

$$|a|n\rangle|^2 = \langle n|a^\dagger a|n\rangle = n\langle n|n\rangle = n = |c|^2 \implies c = \sqrt{n}. \tag{41}$$

Therefore the lowering operator a does the following:

$$a|n\rangle = \sqrt{n}|n-1\rangle. \tag{42}$$

As a consequence of 42, we see that the ground state, $|0\rangle$, will be annihilated by the operator a

$$a|0\rangle = 0. \tag{43}$$

Let's repeat for $a^\dagger$:

$$Ha^\dagger|n\rangle = \left(a^\dagger H + [H, a^\dagger]\right)|n\rangle = \left(a^\dagger H + a^\dagger \hbar\omega\right)|n\rangle = \hbar\omega\left(n + 1 + \frac{1}{2}\right)a|n\rangle, \tag{44}$$

which shows that the state $a^\dagger|n\rangle$ can be indexed as $n+1$, i.e., $a^\dagger|n\rangle = d|n+1\rangle$ where d is the normalization constant. The overall coefficient d can be calculated as

$$|a^\dagger|n\rangle|^2 = \langle n|aa^\dagger|n\rangle = \langle n|a^\dagger a + [a, a^\dagger]|n\rangle = (n+1)\langle n|n\rangle = n = |d|^2 \implies d = \sqrt{n+1}. \tag{45}$$

Therefore the lowering operator $a^\dagger$ does the following:

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{46}$$

By recursively applying $a^\dagger$ on $|0\rangle$ we can get the n-th energy eigenstate, $|n\rangle$:

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle. \tag{47}$$

Let's take a look at certain expectation values. We can immediately see that the expected values of x and $\hat{p} = i\hbar\frac{d}{dx}$ vanish since the integrands of $\langle\psi_n|x|\psi_n\rangle$ and $\langle\psi_n|\hat{p}|\psi_n\rangle$ are odd and the integration range is symmetric around the origin. Equivalently we can do the computation using the ladder operators:

$$\langle\hat{x}\rangle_n = \langle n|\hat{x}|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle n|(a + a^\dagger)|n\rangle = 0. \tag{48}$$

Similarly for p, we have:

$$\langle\hat{p}\rangle_n = \langle n|\hat{p}|n\rangle = -i\sqrt{\frac{m\hbar\omega}{2}}\langle n|a - a^\dagger|n\rangle = 0. \tag{49}$$

Now consider the quadratic operators:

$$\langle\hat{x}^2\rangle_n = \langle n|\hat{x}^2|n\rangle = \frac{\hbar}{2m\omega}\langle n|(a + a^\dagger)^2|n\rangle = \frac{\hbar}{2m\omega}\langle n|(2a^\dagger a + [a, a^\dagger])|n\rangle = \frac{\hbar}{2m\omega}(2n+1). \tag{50}$$

Similarly for p, we have:

$$\langle\hat{p}^2\rangle_n = \langle n|\hat{p}^2|n\rangle = -\frac{m\hbar\omega}{2}\langle n|(a-a^\dagger)^2|n\rangle = \frac{m\hbar\omega}{2}\langle n|2a^\dagger a + [a,a^\dagger]|n\rangle = \frac{\hbar}{2m\omega}(2n+1). \quad (51)$$

The uncertainity in x and p for state n are given as:

$$\langle(\Delta x)^2\rangle_n = \langle\hat{x}^2\rangle_n - (\langle\hat{x}\rangle_n)^2 = \frac{\hbar}{2m\omega}(2n+1), \quad (52)$$

and

$$\langle(\Delta p)^2\rangle_n = \langle\hat{p}^2\rangle_n - (\langle\hat{p}\rangle_n)^2 = \frac{\hbar m\omega}{2}(2n+1). \quad (53)$$

The Heisenberg relation becomes

$$(\Delta x)^2(\Delta p)^2 = \frac{\hbar^2}{4}(2n+1)^2, \quad (54)$$

which has the minimum value of $\frac{\hbar^2}{4}$ at $n = 0$.

Assume that there exists a state $|\alpha\rangle$ which is an eigenstate of the lowering operator a:

$$a|\alpha\rangle = \alpha|\alpha\rangle, (\#eq : aes) \quad (55)$$

where α is a complex number we will need to calculate.

We can quickly compute the expected values of $\hat{x}$ and $\hat{p}$:

$$\langle\hat{x}\rangle_\alpha = \sqrt{\frac{\hbar}{2m\omega}}\langle a+a^\dagger\rangle_\alpha = \sqrt{\frac{\hbar}{2m\omega}}(\alpha+\alpha^*) = \sqrt{\frac{\hbar}{m\omega}}\sqrt{2}\Re(\alpha)(\#eq : xexpect), \quad (56)$$

and

$$\langle\hat{p}\rangle_\alpha = -i\sqrt{\frac{m\hbar\omega}{2}}\langle a-a^\dagger\rangle_\alpha = -i\sqrt{\frac{m\hbar\omega}{2}}(\alpha-\alpha^*) = \sqrt{m\hbar\omega}\sqrt{2}\Im(\alpha).(\#eq : pexpect) \quad (57)$$

And the expected values of $\hat{x}^2$ and $\hat{p}^2$ are:

$$\langle\hat{x}^2\rangle_\alpha = \frac{\hbar}{2m\omega}\langle(a+a^\dagger)^2\rangle_\alpha = \frac{\hbar}{2m\omega}[(\alpha+\alpha^*)^2+1](\#eq : xsqexpect), \quad (58)$$

and

$$\langle\hat{p}^2\rangle_\alpha = -\frac{m\hbar\omega}{2}\langle(a-a^\dagger)^2\rangle_\alpha = \frac{m\hbar\omega}{2}[1-(\alpha-\alpha^*)^2](\#eq : psqexpect) \quad (59)$$

The uncertainity relations become

$$\begin{aligned}\langle(\Delta x)^2\rangle_\alpha &= \langle\hat{x}^2\rangle_\alpha - (\langle\hat{x}\rangle_\alpha)^2 = \frac{\hbar}{2m\omega},\\ \langle(\Delta p)^2\rangle_n &= \langle\hat{p}^2\rangle_\alpha - (\langle\hat{p}\rangle_\alpha)^2 = \frac{\hbar m\omega}{2},\\ \Delta x\Delta p &= \frac{\hbar}{2}(\#eq:cdeltaxdeltap). \end{aligned} \tag{60}$$

$\frac{\hbar}{2}$ is equal to uncertainty value for the ground state.

Since the energy eigenstates, $|n\rangle$, form a complete basis, we should be able to express any state as a combination of $|n\rangle$'s as follows:

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n|n\rangle(\#eq:alphas). \tag{61}$$

We can isolate the coefficients by projecting the sum onto state $\langle m|$ and use the orthoganality of the basis vectors to collapse the summation:

$$\langle m|\alpha\rangle = \sum_{n=0}^{\infty} c_n\langle m|n\rangle = \sum_{n=0}^{\infty} c_n\delta_{m,n} = c_m \implies c_m = \langle m|\alpha\rangle(\#eq:alphascm2). \tag{62}$$

Furthermore, we can express $\langle m|$ as $\left(\frac{(a^\dagger)^m}{\sqrt{m!}}|0\rangle\right)^\dagger = \langle 0|\frac{a^m}{\sqrt{m!}}$ Inserting this back into Eq. @ref(eq:alphas), we get

$$|\alpha\rangle = \sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{n!}}\langle 0|a^n|\alpha\rangle\right)|n\rangle = \sum_{n=0}^{\infty}\left(\langle 0|\alpha\rangle\frac{\alpha^n}{\sqrt{n!}}\right)|n\rangle(\#eq:alphascm). \tag{63}$$

For $|\alpha\rangle$ to be normalized we will need the sum of the squares of the terms in the parenthesis to be unity:

$$\sum_{n=0}^{\infty}|\langle 0|\alpha\rangle|^2\frac{|\alpha|^{2n}}{n!} = |\langle 0|\alpha\rangle|^2 e^{|\alpha|^2} = 1 \implies \langle 0|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}(\#eq:alphanorm). \tag{64}$$

Hence the final form of the coherent state $|\alpha\rangle$ is

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle(\#eq:alphafinal). \tag{65}$$

$|\alpha\rangle$ is a superposition of energy eigenstates and we can compute the expected value of n by using the number operator N:

$$\begin{aligned}
\lambda &\equiv \langle\alpha|N|\alpha\rangle = \langle\alpha|a^\dagger a|\alpha\rangle = e^{-|\alpha|^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha^*)^m\alpha^n}{\sqrt{n!}\sqrt{m!}}\langle m|a^\dagger a|n\rangle = e^{-|\alpha|^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha^*)^m\alpha^n}{\sqrt{n!}\sqrt{m!}}n\delta_{m,n} \\
&= e^{-|\alpha|^2}\sum_{n=0}^{\infty}n\frac{|\alpha|^{2n}}{n!} = e^{-|\alpha|^2}\sum_{n=0}^{\infty}\left[\beta\frac{d}{d\beta}\frac{\beta^n}{n!}\right]_{\beta=\alpha^2} = e^{-|\alpha|^2}\left[\beta\frac{d}{d\beta}e^{\beta}\right]_{\beta=|\alpha|^2} = |\alpha|^2(\#eq:nexpect), \quad (66)
\end{aligned}$$

We can use the expected value of n, which we defined as λ to prescribe the probability distribution of n:

$$P(n) = \frac{\lambda^n}{n!}e^{-\lambda}(\#eq:pon), \quad (67)$$

which is nothing but the density function of the Poisson distribution. Let's check for the orthogonality of the coherent states

$$\langle\alpha|\alpha'\rangle = e^{-\frac{1}{2}|\alpha|^2}e^{-\frac{1}{2}|\alpha'|^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\alpha^{*m}}{\sqrt{m!}}\frac{\alpha'^n}{\sqrt{n!}}\underbrace{\langle m|n\rangle}_{\delta_{m,n}} = e^{(-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\alpha'|^2)}e^{\alpha^*\alpha'} = e^{-\frac{1}{2}|\alpha-\alpha'|^2}(\#eq:ortho), \quad (68)$$

which shows that coherent states are not orthogonal. Alhough they are not mutually orthogonal, the set of coherent states are complete. Keep in mind that α is a complex number and it can be represented as $re^{i\theta}$. We can sweep through all the coherent states by sweeping through the parameters r and θ. Consider the following operator:

$$\begin{aligned}
\int_0^\infty drr\int_0^{2\pi}d\theta|\alpha\rangle\langle\alpha| &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{\sqrt{m!}}\frac{1}{\sqrt{n!}}|n\rangle\langle m|\int_0^\infty drre^{-r^2}r^{m+n}\underbrace{\int_0^{2\pi}d\theta e^{i\theta(n-m)}}_{2\pi\delta_{m,n}} \\
&= \sum_{n=0}^{\infty}\frac{\pi}{n!}|n\rangle\langle n|\underbrace{\int_0^\infty d(r^2)re^{-r^2}(r^2)^n}_{n!} = \pi\sum_{n=0}^{\infty}|n\rangle\langle n| = \pi I.(\#eq:inte) \quad (69)
\end{aligned}$$

This shows that with proper normalization, we can create a unity operator using the coherent states:

$$\frac{1}{\pi}\int d^2\alpha|\alpha\rangle\langle\alpha| = I,(\#eq:unitop) \quad (70)$$

where $d^2\alpha$ stands for $rdrd\theta$.

We can further compactify the representation of the coherent state given in Eq. @ref(eq:alphafinal) by exponentiating the raising operator:

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle = e^{-\frac{1}{2}|\alpha|^2}e^{\alpha a^\dagger}|0\rangle(\#eq:alphafinal2). \tag{71}$$

We can project $|\alpha\rangle$ onto $|x\rangle$ to find how the coherent state looks in the coordinate representation:

$$\langle x|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\langle x|e^{\alpha\sqrt{\frac{m\omega}{2\hbar}}\left(x-\frac{1}{m\omega}\frac{d}{dx}\right)}|0\rangle = e^{-\frac{1}{2}|\alpha|^2}e^{\alpha\sqrt{\frac{m\omega}{2\hbar}}\left(x-\frac{1}{m\omega}\frac{d}{dx}\right)}\langle x|0\rangle(\#eq:coordinaterep), \tag{72}$$

where

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}e^{-\frac{m\omega x^2}{2\hbar}}(\#eq:coherentx0). \tag{73}$$

The exponentiated operator is rather nontrivial to compute. Let's address that in a generic setting. Consider the following expression with two operators A and B:

$$\begin{aligned} e^{A+B} &= 1+A+B+\frac{1}{2}\left(A^2+B^2+AB+BA\right)+\cdots \\ &= \left(1+A+\frac{A^2}{2}+\cdots\right)\left(1+B+\frac{B^2}{2}+\cdots\right)\left(1-\frac{1}{2}[A,B]+\cdots\right) \\ &= e^Ae^Be^{-\frac{1}{2}[A,B]}(\#eq:exponentiated). \end{aligned} \tag{74}$$

In our current problem, we have $A=\alpha\sqrt{\frac{m\omega}{2\hbar}}x$ and $B=-\alpha\sqrt{\frac{\hbar}{2m\omega}}\frac{d}{dx}$, which gives $[A,B]=\frac{\alpha^2}{2}$.

$$\langle x|\alpha\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}e^{-\frac{1}{2}|\alpha|^2}e^{\alpha\sqrt{\frac{m\omega}{2\hbar}}\left(x-\frac{1}{m\omega}\frac{d}{dx}\right)}e^{-\frac{m\omega x^2}{2\hbar}} = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}e^{-\frac{1}{2}|\alpha|^2}e^{-\frac{\alpha^2}{4}}e^{\alpha\sqrt{\frac{m\omega}{2\hbar}}x}e^{-\alpha\sqrt{\frac{\hbar}{2m\omega}}\frac{d}{dx}}e^{-\frac{m\omega x^2}{2\hbar}}(\#eq:coherent$$

Let's deal with the exponentiated derivative in a generic sense as follows:

$$e^{\kappa\frac{d}{dx}}f(x) = \sum_{n=0}^{\infty}\frac{\kappa^n}{n!}\frac{d^n}{dx^n}f(x) = f(x+\kappa),(\#eq:translate) \tag{76}$$

where we observed that the series nothing but the Taylor expansion of $f(x+\kappa)$. This is no surprise since $e^{\kappa\frac{d}{dx}}$ is nothing but the translation operator! Putting this back in we get:

$$\begin{aligned}
\langle x|\alpha\rangle &= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} exp\left(-\frac{1}{2}|\alpha|^2 - \frac{\alpha^2}{4} + \alpha\sqrt{\frac{m\omega}{2\hbar}}x - \frac{m\omega\left(x-\alpha\sqrt{\frac{\hbar}{2m\omega}}\right)^2}{2\hbar}\right) \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} exp\left(-\frac{1}{2}|\alpha|^2 - \frac{\alpha^2}{2} + \sqrt{2}\alpha\sqrt{\frac{m\omega}{\hbar}}x - \frac{m\omega}{2\hbar}x^2\right) \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} exp\left(-\frac{1}{2}\left(\sqrt{\frac{m\omega}{\hbar}}x - \sqrt{2}\Re(\alpha)\right)^2 + i\sqrt{2}\sqrt{\frac{m\omega}{\hbar}}x\Im(\alpha) - i\Im(\alpha)\Re(\alpha)\right) \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} exp\left(-\frac{m\omega}{2\hbar}\left(x-\langle x\rangle_\alpha\right)^2 + \frac{i}{\hbar}\langle p\rangle_\alpha x - \frac{i}{2\hbar}\langle p\rangle_\alpha\langle x\rangle_\alpha\right)(\#eq:coherentx3)(77)
\end{aligned}$$

Time evolution

We can compute the time dependence of the coherent states by acting on $|\alpha\rangle$ by the time development operator:

$$U(t) = e^{-\frac{i}{\hbar}Ht}(\#eq:tdev), \tag{78}$$

which gives

$$|\alpha, t\rangle = U(t)|\alpha\rangle = e^{-\frac{i}{\hbar}Ht}|\alpha\rangle(\#eq:alphat), \tag{79}$$

where $|\alpha, t\rangle$ is the time dependent coherent state. Expanding the coherent state in the energy eigenbasis, we get:

$$\begin{aligned}
|\alpha(t)\rangle &= U(t)|\alpha\rangle = e^{-\frac{i}{\hbar}Ht}|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}e^{-\frac{iHt}{\hbar}}|n\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}e^{-i\omega t(n+\frac{1}{2})}|n\rangle \\
&= e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}e^{-i\omega t(n+\frac{1}{2})}\frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle = e^{-\frac{1}{2}|\alpha|^2}e^{-i\frac{\omega t}{2}}\sum_{n=0}^{\infty}\frac{\left(\alpha e^{-i\omega t}a^\dagger\right)^n}{n!}|0\rangle \\
&= e^{-i\frac{\omega t}{2}}\left(e^{-\frac{1}{2}|\alpha|^2}e^{\alpha e^{-i\omega t}a^\dagger}|0\rangle\right) = e^{-i\frac{\omega t}{2}}|e^{-i\omega t}\alpha\rangle, (\#eq:alphat2) \qquad (80)
\end{aligned}$$

where we used Eq. @ref(eq:alphafinal2). We can drop the overall phase factor to write:

$$\alpha(t) = e^{-i\omega t}\alpha(\#eq:alphatfinale). \tag{81}$$

In order to compute $\langle x\rangle_{\alpha(t)}$, let's first write $\alpha = |\alpha|e^{i\sigma}$, where σ is the initial phase. Then we have

$$\langle\hat{x}\rangle_{\alpha(t)} = \sqrt{\frac{\hbar}{m\omega}}\sqrt{2}\Re(\alpha(t)) = \sqrt{\frac{2\hbar}{m\omega}}|\alpha|\cos(\sigma-\omega t)(\#eq:xexpecttd). \tag{82}$$

Similarly, the expected value of $\hat{p}$ reads:

$$\langle\hat{p}\rangle_{\alpha(t)} \quad = \quad \sqrt{m\hbar\omega}\sqrt{2}\Im(\alpha) = \sqrt{2m\hbar\omega}|\alpha|\sin(\sigma-\omega t)(\#eq:pexpecttd). \tag{83}$$

Equations @ref(eq:xexpecttd) and @ref(eq:pexpecttd) show that the expected values of x and p in the coherent state look just like the classical harmonic oscillator.

Finally, let's compute the explicit form of the wave function with its time dependence. Al we need to do is to transform $\alpha \to e^{-i\omega t}\alpha$ and $|\alpha\rangle \to e^{-i\frac{\omega t}{2}}|e^{-i\omega t}\alpha\rangle$ and put these back in Eq. @ref(eq:coherentx3) to get

$$\psi_\alpha(x,t) = \langle x|\alpha\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} exp\left(-\frac{m\omega}{2\hbar}\left(x-\langle x\rangle_{\alpha(t)}\right)^2 + \frac{i}{\hbar}\langle p\rangle_{\alpha(t)}x - i\frac{\omega t}{2} + i\frac{|\alpha|^2}{2}\sin\left(2\omega t - 2\sigma\right)\right)(\#eq:psifinal \tag{84}$$

[1] G. B. Arfken and H. J. Weber, *Mathematical Methods for Physicists, 4th edition.* Academic Press, San Diego, 1995, pp. 537–547.