

# Anisotropies in the Gravitational-Wave Stochastic Background

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We consider anisotropies in the stochastic background of gravitational-waves (SBGW) arising from random fluctuations in the number of gravitational-wave sources. We first develop the general formalism which can be applied to different cosmological or astrophysical scenarios. We then apply this formalism to calculate the anisotropies of SBGW associated with the fluctuations in the number of cosmic string loops, considering both cosmic string cusps and kinks. We calculate the anisotropies as a function of angle and frequency.

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A stochastic background of gravitational-wave (SBGW) radiation is produced by a large number of weak, independent and unresolved gravitational wave sources. The sources of the SBGW can be isotropic or anisotropic. For the case of sources of cosmological origin [1–3] the distribution of the gravitational wave sources is expected to be isotropic, while astrophysical sources such as rotating neutron stars [4] or magnetars [5] may have an anisotropic distribution. Even in the case of an a priori isotropic source distribution, random fluctuations in the number of sources will (in general) give rise to anisotropies. Such anisotropies are analogous to the anisotropies observed in the cosmic microwave background (CMB) radiation and would carry additional information about the gravitational-wave sources that generated them.

Cosmic strings are expected to contribute to the SBGW. Cosmic strings are predicted by a large class of unified theories [6–8] as remnants of spontaneously broken symmetries at phase transitions in the early Universe, as well as in string-theory-inspired cosmological scenarios [9]. Once formed, a network of cosmic strings evolves toward an attractor solution called the scaling regime, in which the statistical properties of the network, such as the average distance between strings and the size of loops at formation, scale with the cosmic time. The gravitational interaction of strings is characterized by the dimensionless parameter  $G\mu$ , where  $G$  is Newton's constant and  $\mu$  is the tension. The current CMB bound on the tension is  $G\mu < 6.1 \times 10^{-7}$  [10, 11]. Cosmic string cusps, regions of string that acquire enormous Lorentz boosts, are expected to generate large transient gravitational-wave signals [12–14]. Such individual bursts could be observable by current and planned gravitational wave detectors [15] for values of  $G\mu$  as low as  $10^{-13}$ , which may provide a probe of a certain class of string theories [9]. A SBGW produced by the incoherent superposition of cusp bursts from a network of cosmic strings and superstrings was considered in [16, 17], and it was later shown that kinks contribute to the SBGW

at the same order as cusps [18]. These sources of SBGW are also observable by current and planned detectors, for a wide range of the parameter space, see [17, 18] and references therein.

In this paper, we develop a general formalism to treat SBGW anisotropies. In particular, we consider two-point correlations in SBGW between two different directions in the sky, which arise from random fluctuations in the number of gravitational-wave sources. While this formalism is applicable to a variety of cosmological and astrophysical SBGW models (see [19] and the references therein), we illustrate it for the specific case of cosmic (super)string cusps and kinks.

*Anisotropies in the SBGW:* We start from formalism in references [13, 14] and extend it to treat angular dependence. The energy density of SBGW at frequency  $f$  corresponding to sources in the direction  $\hat{\Omega}$  is given by

$$\Omega_{\text{gw}}(f, \hat{\Omega}) \equiv \frac{f}{\rho_c} \frac{d\rho_{\text{gw}}(\hat{\Omega})}{df}, \quad (1)$$

where  $d\rho_{\text{gw}}$  is the energy density of gravitational waves in the frequency range  $f$  to  $f + df$  and  $\rho_c$  is the critical energy density of the Universe. Let us assume that sources are characterized by a set of parameters  $\zeta$  - in the case of cosmic strings, redshift  $z$  is one such parameter. Therefore  $\Omega_{\text{gw}}$  is an integral over the parameter space  $\zeta$ , which we propose to discretize as follows:

$$\begin{aligned} \Omega_{\text{gw}}(f, \hat{\Omega}) &= \int d\zeta n(\zeta, \hat{\Omega}) w(f, \zeta, \hat{\Omega}) \\ &\simeq \sum_i \Delta(\zeta_i) n(\zeta_i, \hat{\Omega}) w(f, \zeta_i, \hat{\Omega}) \\ &\equiv \sum_i N(\zeta_i, \hat{\Omega}) w(f, \zeta_i, \hat{\Omega}). \end{aligned} \quad (2)$$

Here we assume that the parameter space can be divided into disjoint volumes  $\Delta(\zeta_i)$ , centered at  $\zeta_i$ , whose size is large compared to the correlation length of the number of sources. In other words, a statistical fluctuation in the number of sources in one volume would have no

implications on the number of sources in any other volume. We further define  $n(\zeta_i, \hat{\Omega})$  as the number density of sources (i.e number per parameter space volume) in the direction  $\hat{\Omega}$  with the parameter set  $\zeta_i$ , and  $w(f, \zeta_i, \hat{\Omega})$  as the contribution to  $\Omega_{\text{gw}}$  of one source at frequency  $f$ , in the direction  $\hat{\Omega}$ , and with the parameter set  $\zeta_i$ . We also define  $N(\zeta_i, \hat{\Omega}) \equiv n(\zeta_i, \hat{\Omega})\Delta(\zeta_i)$  as the total number of sources with the parameters in the range from  $\zeta_i$  to  $\zeta_i + \Delta(\zeta_i)$  and in the direction  $\hat{\Omega}$ . The contribution of one source is given by

$$w(f, \zeta_i, \hat{\Omega}) \equiv \frac{4\pi^2 f^3}{3H_0^2} h^2(f, \zeta_i, \hat{\Omega}) \mathcal{R}(f, \zeta_i, \hat{\Omega}), \quad (3)$$

where  $h(f, \zeta_i, \hat{\Omega})$  is the strain of the gravitational wave with frequency  $f$  originating from a source with parameters  $\zeta_i$  and at the line of sight  $\hat{\Omega}$ .  $\mathcal{R}(f, \zeta_i, \hat{\Omega})$  represents the observable part of the gravitational radiation from the source, i.e. it incorporates the propagation of the wave in the expanding universe as well as possible beaming effects, see [12, 13].

The angular dependence of  $N$  can originate from anisotropic source distribution. Moreover, even in the case of an a priori isotropic SBGW, random fluctuations in the number of sources will (in general) give rise to anisotropies. Note that  $N(\zeta_i, \hat{\Omega})$  are dimensionless numbers, which are by construction uncorrelated for different values of the index  $i$ . Assuming Poisson distribution, the statistical fluctuations of  $N(\zeta_i, \hat{\Omega})$  are of order  $\sqrt{N(\zeta_i, \hat{\Omega})}$ . The corresponding fluctuation in  $\Omega_{\text{gw}}$  is

$$\delta\Omega_{\text{gw}}(f, \hat{\Omega}) = \sum_i \delta N(\zeta_i, \hat{\Omega}) w(f, \zeta_i, \hat{\Omega}), \quad (4)$$

The two-point correlation of  $\delta\Omega_{\text{gw}}(f, \hat{\Omega})$  at two different directions reads

$$\begin{aligned} \mathcal{C} &\equiv \left\langle \delta\Omega_{\text{gw}}(f, \hat{\Omega}) \delta\Omega_{\text{gw}}(f, \hat{\Omega}') \right\rangle \\ &= \sum_{i,j} w(f, \zeta_i, \hat{\Omega}) w(f, \zeta_j, \hat{\Omega}') \left\langle \delta N(\zeta_i, \hat{\Omega}) \delta N(\zeta_j, \hat{\Omega}') \right\rangle. \end{aligned} \quad (5)$$

Since the fluctuations in the number of gravitational-wave sources are Poissonian, we propose the following bilinear expectation:

$$\left\langle \delta N(\zeta_i, \hat{\Omega}) \delta N(\zeta_j, \hat{\Omega}') \right\rangle \sim N(\zeta_i, \hat{\Omega}) \mathcal{F}(\gamma, \zeta_i) \delta_{ij}, \quad (6)$$

where  $\gamma$  is the angle between  $\hat{\Omega}$  and  $\hat{\Omega}'$ .  $\mathcal{F}$  is a function that incorporates the correlation properties of the gravitational wave sources. Although the precise form of  $\mathcal{F}$  will depend on the problem at hand, we can discuss several properties of this function. Firstly, we expect to see the maximum correlation if the two sources are close to each other in the physical space as well as the parameter space. Therefore  $\mathcal{F}$  must assume its maximum value at

$\gamma = 0$ , and it should decrease for larger values of  $\gamma$ . Since  $\mathcal{F}$  constrains  $\gamma$ , the angle between  $\hat{\Omega}$  and  $\hat{\Omega}'$ , to small values, we keep only  $\hat{\Omega}$  at the right hand side of Eq. (6). This is a good approximation as long as  $N$  changes slowly with  $\hat{\Omega}$ . Below we will consider an explicit example and discuss form of  $\mathcal{F}$  in more detail. Inserting this into Eq. (5) gives

$$\begin{aligned} \mathcal{C} &= \sum_i \Delta(\zeta_i) n(\zeta_i, \hat{\Omega}) w^2(f, \zeta_i, \hat{\Omega}) \mathcal{F}(\gamma, \zeta_i) \\ &\rightarrow \int d\zeta n(\zeta, \hat{\Omega}) w^2(f, \zeta, \hat{\Omega}) \mathcal{F}(\gamma, \zeta), \end{aligned} \quad (7)$$

where we take the integral limit of the sum. This is a general expression applicable to both cosmological and astrophysical problems in which the correlation properties of the sources are specified by the function  $\mathcal{F}$ .

*Cosmic Strings Case:* We now apply this formalism to the case of cosmic strings, including gravitational-wave bursts from cusps and kinks, in which the distribution of sources is specified by the redshift  $z$ . It is convenient to parameterize physical quantities in terms of redshift. To this end, we define the following dimensionless cosmological functions:

$$\begin{aligned} \varphi_r(z) &= \int_0^z \frac{dz'}{\mathcal{H}(z')}, \\ \varphi_t(z) &= \int_z^\infty \frac{dz'}{(1+z')\mathcal{H}(z')}, \\ \varphi_V(z) &= \frac{\varphi_r^2(z)}{(1+z)^3 \mathcal{H}(z)}, \end{aligned} \quad (8)$$

where  $\mathcal{H}(z) = \sqrt{\Omega_m(1+z)^3 + \Omega_r(1+z)^4 + \Omega_\Lambda}$  is the Hubble function with  $\Omega_M = 0.25$ ,  $\Omega_R = 4.6 \times 10^{-5}$  and  $\Omega_\Lambda = 1 - \Omega_M - \Omega_R$ . We now explicitly construct the integral in Eq. (7). Firstly, the parameter space volume  $d\zeta$  in this case is simply the co-moving volume. It can be written as  $H_0^{-3} \varphi_V(z) dz$ , where  $H_0$  is the present value of the Hubble constant. This converts the co-moving differential volume  $r^2 dr$  to the corresponding differential volume as a function of the redshift. The next quantity in Eq. (7) is the number density of the loops. If the loop size is determined by gravitation back-reaction [12, 13], the loop number density is given by

$$n(z) \approx \frac{c(z)}{p \Gamma G \mu t^3(z)}, \quad (9)$$

where  $p$  is the reconnection probability,  $\Gamma = 50$  is a dimensionless parameter proportional to the power emitted in gravitational waves by cosmic string loops, and  $t(z)$  is the cosmic time which can be written as  $t(z) = \varphi_t(z)/H_0$ . The function  $c(z) \equiv 1 + \frac{9z}{z+z_{\text{eq}}}$  ( $z_{\text{eq}} \simeq 5440$ ) accounts for the fact that the loop density in radiation domination is about 10 times that of the matter domination. In order to define the  $\mathcal{F}$ -function in Eq. (7), we assume that

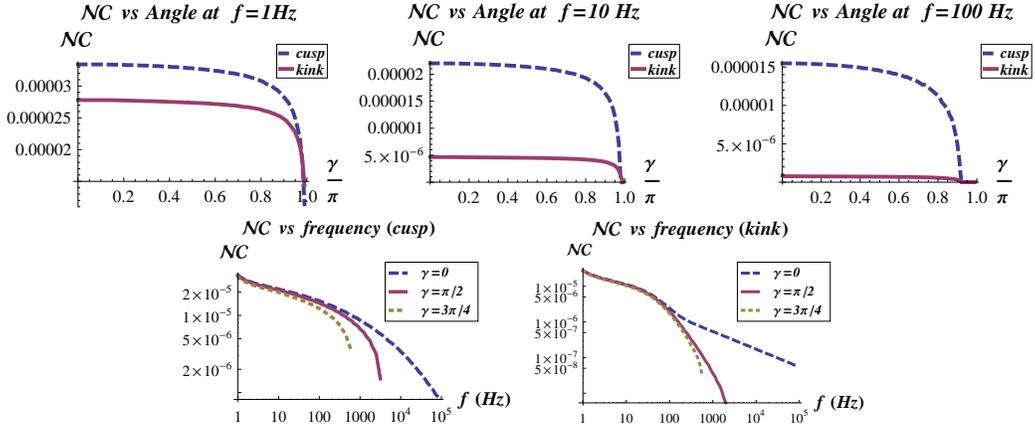


FIG. 1: Top: Normalized Correlation  $\mathcal{NC}$  as functions of  $\gamma/\pi$  for cusps and kinks for  $f = 1\text{Hz}$ ,  $f = 10\text{Hz}$  and  $f = 100\text{Hz}$ , for  $G\mu = 1.0 \times 10^{-8}$ ,  $p = 1$  and  $\epsilon = 1.0 \times 10^{-11}$ . Bottom:  $\mathcal{NC}$  as functions of frequency for cusps and kinks, for various values of  $\gamma$  and for the same model parameters as above.

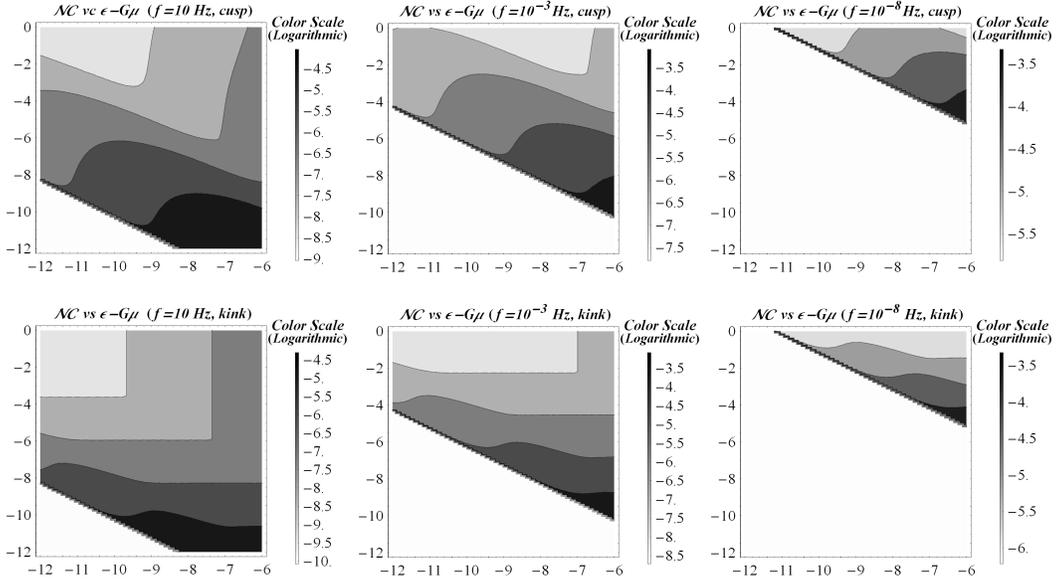


FIG. 2:  $\mathcal{NC}$  for cusps and kinks in  $\text{Log}(\epsilon)$  (vertical axis)  $\text{Log}(G\mu)$ (horizontal axis) parameter space at frequencies applicable to ground-based detectors (10 Hz) [15], satellite-based detectors (1 mHz) [21], and pulsar-based observations ( $10^{-8}$  Hz) [22]. The (base 10 logarithm of) numerical values of  $\mathcal{NC}$  are denoted in the color bar for each plot.

the number density of cosmic string cusps and kinks at a given  $z$  is correlated over the length scale  $R(z)$  given by the Hubble size,  $R(z) \approx t(z)$  [23]. The angular size spanned by this length scale at the distance  $r(z)$  can be calculated using the standard *angular diameter-redshift relation* as

$$\gamma_z = 2 \arctan \left[ \frac{(1+z)R(z)}{r(z)} \right], \quad (10)$$

where  $r(z)$  can be written as  $r(z) = \varphi_r(z)/H_0$ . Therefore, for the given redshift  $z$ , two directions on the sky are correlated if their angular separation,  $\gamma$ , is less than

$\gamma_z$ . This condition can be imposed by the  $\mathcal{F}$  function,

$$\mathcal{F}(\gamma, z) \equiv \Theta \left[ 1 - \frac{\gamma}{\gamma_z} \right], \quad (11)$$

which vanishes if  $\gamma$  is larger than  $\gamma_z$ , the angle subtended by the length scale  $R(z)$ . We emphasize that the correlations considered here are large scale, and arise from the fluctuations in the number of cosmic string loops in an evolving cosmic string network. This is different from the correlations associated with the correlation length of a single cosmic string loop, which are important in determining the cosmic string signatures in the CMB [20]. For cusps and kinks on cosmic string loops with sizes given

by the gravitational back-reaction scale we have [18]

$$w_c(f, z) = \frac{2\pi^2(G\mu)^2}{3(1+z)^{7/3}\varphi_r^2\varphi_t^{1/3}} \frac{\Theta\left[1 - \left(\frac{\alpha f}{H_0}(1+z)\varphi_t\right)^{-1}\right]}{\left(\frac{\alpha f}{H_0}\right)^{1/3}}$$

$$w_k(f, z) = \frac{4\pi^2(G\mu)^2}{3(1+z)^{8/3}\varphi_r^2\varphi_t^{2/3}} \frac{\Theta\left[1 - \left(\frac{\alpha f}{H_0}(1+z)\varphi_t\right)^{-1}\right]}{\left(\frac{\alpha f}{H_0}\right)^{2/3}}, \quad (12)$$

where  $\alpha \equiv \epsilon\Gamma G\mu$  is the parameter that sets the length of the loops. Since the function  $w$  has no angle dependence for kinks and cups, Eq. (7) simplifies to

$$\mathcal{C} = \mathcal{C}(f, \gamma) = \int dz H_0^{-3} \varphi_V(z) n(z) w^2(f, z) \mathcal{F}(\gamma, z)$$

$$= \int dz \varphi_V(z) \frac{c(z)(p\Gamma G\mu)^{-1}}{\varphi_t^3(z)} w^2(f, z) \Theta\left[1 - \frac{\gamma}{\gamma_z}\right] \quad (13)$$

which is a function of the opening angle,  $\gamma$ , and the frequency only. It is important to note that large rare events which occur at rates smaller than the relevant time-scale of the experiment are excluded [12] from  $\mathcal{C}$  in numerical calculation. This exclusion removes the a priori divergence of the integrand at  $z = 0$ . The integrand of Eq. (13) quickly vanishes with increasing redshift, implying that the dominant contribution comes from low redshifts. The small values of redshift correspond to closer sources, which have larger angular size in the sky. Therefore the angular dependence of correlations will be rather flat for small angles, and it will rapidly vanish for large angles, for which small values of redshift are excluded from the integral by  $\mathcal{F}(\gamma, z)$ . In order to understand the relative strength of the fluctuations at a given frequency  $f$  compared to  $\Omega_{\text{gw}}(f)$  (integrated over all sky) we define the following quantity:  $\mathcal{N}\mathcal{C}(f, \gamma) \equiv \frac{\sqrt{\mathcal{C}(f, \gamma)}}{\Omega_{\text{gw}}(f)}$ , which we refer to as the *normalized correlation*. We numerically evaluate the integrals in Eq. (13) for kinks and cusps and calculate the normalized correlations, as depicted in Fig. 1. We also do a parameter scan in  $\epsilon - G\mu$  space. Fig. 2 shows the density plot for the strength of the background  $\Omega_{\text{gw}}$  and  $\mathcal{N}\mathcal{C}$  at various values of  $f$  for cusps and kinks.

*Conclusions:* In this paper we have developed the formalism for calculating the spatial anisotropies in the stochastic background of gravitational waves associated with the random fluctuations in the number of sources. The formalism is applicable to a variety of cosmological and astrophysical models. We applied it to the case of SBGW due to cosmic (super)string cusps and kinks, and observed that the relative strength of the anisotropies,  $\sqrt{\mathcal{C}}/\Omega_{\text{gw}}$ , can be estimated by  $1/\sqrt{N} = \sqrt{\Gamma G\mu}$ , which can be as high as  $10^{-3}$ . While observation of these spatial anisotropies is unlikely for the second-generation detectors that are currently being built (Advanced LIGO and Advanced Virgo), the planned third-generation detectors

(such as Einstein Telescope) should be sufficiently sensitive to measure them over a large part of the parameter space. We emphasize that the general formalism developed here can be used to distinguish between different SBGW models - that is, between models that predict similar frequency spectra and different spatial anisotropies. This technique will be crucial for the identification of the source of SBGW which is expected to be observed by the future generations of the gravitational-wave detectors.

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