Scattering fermions and scalars

2025-04-10

We present a detailed calculation of scalar-fermion scattering via Yukawa interactions. Starting from the Lagrangian with a $\phi\bar{\psi}\psi$ coupling, we derive the Feynman diagrams and their corresponding amplitudes. We evaluate these amplitudes explicitly by calculating the s-channel and u-channel contributions, and demonstrate how to square them to obtain the differential cross-section.

blog: https://tetraquark.vercel.app/posts/scalar_spinor/

email: quarktetra@gmail.com

Lagrangian and Feynman Diagrams

We would like to compute the cross section of fermion-boson scattering process. The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{m^2}{2}\phi^2 + i\bar{\psi}\gamma^\mu\partial_\mu\psi - M\bar{\psi}\psi + h\phi\bar{\psi}\psi - \frac{\lambda}{4!}\phi^4, \tag{1}$$

where ϕ represents the neutral scalar particle, and ψ_α is a four-component spinor field with $\alpha = 1, 2, 3, 4$. The scattering process we are after is given as

$$\phi(k_1) + \psi(p_1) \longrightarrow \phi(k_2) + \psi(p_2). \tag{2}$$

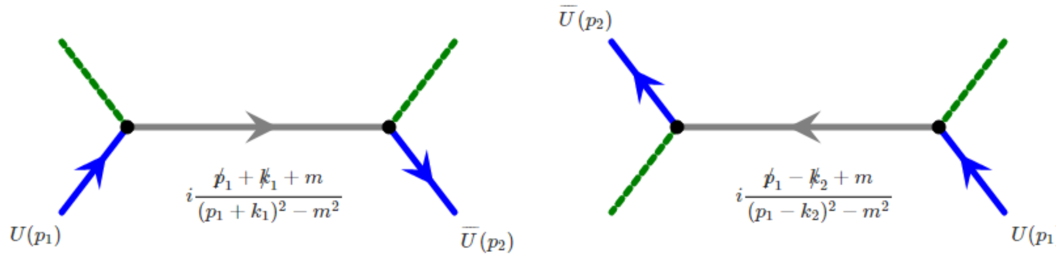

Figure 1: Two Feynman diagrams, with amplitudes $\mathcal{M}_A$ and $\mathcal{M}_B$, contributing to the scattering. Hover on the lines and vertices to see more info.

Amplitudes

The amplitudes for the diagrams in Figure 1 can be written as

$$\begin{aligned} \mathcal{M}_A &= -i\overline{U}(p_2)(-ih)\left[i\frac{\not{p}_1 + \not{k}_1 + M}{(p_1 + k_1)^2 - M^2}\right](-ih)U(p_1) \\ \mathcal{M}_B &= -i\overline{U}(p_2)(-ih)\left[i\frac{\not{p}_1 - \not{k}_2 + M}{(p_1 - k_2)^2 - M^2}\right](-ih)U(p_1). \end{aligned} \tag{3}$$

The numerators can be simplified by using the equation of motion for the fermions, namely:

$$(\not{p}_1 - M)U(p_1) = 0. \tag{4}$$

Let's compute the denominators for the propagators:

$$\begin{aligned} (p_1 + k_1)^2 - M^2 &= p_1^2 + k_1^2 + 2p_1 \cdot k_1 - M^2 = M^2 + m^2 + 2p_1 \cdot k_1 - M^2 \\ &= 2p_1 \cdot k_1 + m^2 \\ (p_1 - k_2)^2 - M^2 &= p_1^2 + k_2^2 - 2p_1 \cdot k_2 - M^2 = M^2 + m^2 - 2p_1 \cdot k_2 - M^2 \\ &= -2p_1 \cdot k_2 + m^2. \end{aligned} \tag{5}$$

Inserting these into Eq. 3, we get

$$\begin{aligned} \mathcal{M}_A &= \frac{-h^2}{2p_1 \cdot k_1 + m^2}\overline{U}(p_2)\left[2M + \not{k}_1\right]U(p_1) \\ \mathcal{M}_B &= \frac{h^2}{2p_1 \cdot k_2 + m^2}\overline{U}(p_2)\left[2M - \not{k}_2\right]U(p_1). \end{aligned} \tag{6}$$

Let's also consider the process in the high energy limit, i.e., $E \gg M, m$, that is we will drop the mass terms. In this limit we can simplify the amplitudes:

$$
\begin{aligned}
\mathcal{M}_A &\simeq \frac{-h^2}{2p_1 \cdot k_1} \overline{U}(p_2) \not{k}_1 U(p_1) \\
\mathcal{M}_B &\simeq \frac{-h^2}{2p_1 \cdot k_2} \overline{U}(p_2) \not{k}_2 U(p_1).
\end{aligned} \tag{7}
$$

Squaring the amplitudes

The total amplitude is given by

$$
\mathcal{M} = \mathcal{M}_A + \mathcal{M}_B. \tag{8}
$$

and we will need to compute its mode-square which will involve mode-squares of the individual amplitudes and the cross terms. We will also average over fermion polarization which will result in trace operations. There are few trace properties of $\gamma-$matrices we will make use of:

$$
\begin{aligned}
\mathrm{Tr}[I] &= 4 \\
\mathrm{Tr}[\gamma^\mu \gamma^\nu] &= 4g^{\mu\nu} && (9) \\
\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] &= 4[g^{\mu\nu} g^{\rho\sigma} + g^{\mu\sigma} g^{\nu\rho} - g^{\mu\rho} g^{\nu\sigma}] && (10) \\
\mathrm{Tr}[\gamma_1^\mu \gamma_2^\mu \cdots \gamma_{2n+1}^\mu] &= 0, && (11)
\end{aligned}
$$

The mode-square of the first amplitude becomes

$$
\begin{aligned}
\left|\overline{\mathcal{M}_A}\right|^2 &= \frac{h^4}{4(p_1 \cdot k_1)^2} \frac{1}{2} \, \mathrm{Tr}\, [\not{p}_2 \not{k}_1 \not{p}_1 \not{k}_1] \\
&= \frac{h^4}{2(p_1 \cdot k_1)^2} p_1 \cdot k_1 \, p_2 \cdot k_2 = h^4 \frac{p_1 \cdot k_2}{p_1 \cdot k_1}.
\end{aligned} \tag{12}
$$

Similarly, the mode-square of the second amplitude reads

$$
\begin{aligned}
\left|\overline{\mathcal{M}_B}\right|^2 &= \frac{h^4}{4(p_1 \cdot k_1)^2} \frac{1}{2} \, \mathrm{Tr}\, [\not{p}_2 \not{k}_2 \not{p}_1 \not{k}_2] \\
&= \frac{h^4}{(p_1 \cdot k_1)^2} p_2 \cdot k_2 \, p_1 \cdot k_2 = h^4 \frac{p_1 \cdot k_1}{p_1 \cdot k_2},
\end{aligned} \tag{13}
$$

where we used conservation of 4−momentum in the last step as follows:

$$
\begin{aligned}
p_1 + k_1 &= p_2 + k_2 \iff p_1 - k_2 = p_2 - k_1 \\
(p_1 + k_1)^2 &= (p_2 + k_2)^2 \implies p_1 \cdot k_1 = p_2 \cdot k_2,
\end{aligned} \tag{14}
$$

Finally one of the cross term can be calculated as

$$\begin{aligned}\overline{\mathcal{M}_A^*\mathcal{M}_B} &= \frac{-h^4}{4p_1\cdot k_1\, p_1\cdot k_2}\frac{1}{2}\,\mathrm{Tr}\,[\not{p}_2\not{k}_1\not{p}_1\not{k}_2] \\ &= \frac{h^4}{2p_1\cdot k_1\, p_1\cdot k_2}\,[p_2\cdot k_1\, p_1\cdot k_2 + p_2\cdot k_2\, p_1\cdot k_1 - p_2\cdot p_1\, k_1\cdot k_2] \\ &= \frac{h^4}{2}\left[\frac{p_1\cdot k_2}{p_1\cdot k_1} + \frac{p_1\cdot k_1}{p_1\cdot k_2} - \frac{p_1\cdot p_2\, k_1\cdot k_2}{p_1\cdot k_1\, p_1\cdot k_2}\right]. \end{aligned} \tag{15}$$

Cross-section

Let's find out which term will have the dominant contribution to the cross-section. To this end, we can treat the problem in the center of mass frame and define:

$$\begin{aligned} k_1 &= (\omega, 0, 0, w) \\ p_1 &= (E, 0, 0, -\omega) \\ k_2 &= (\omega, \omega\sin\theta, 0, \omega\cos\theta) \\ p_2 &= (E, 0, 0, -\omega). \end{aligned} \tag{16}$$

We can observe that the term $1/p_1 \cdot k_2$ will be $\sim 1/M^2$ at $\theta = \pm\pi$, and therefore will be the dominating term, since other terms will will behave as $1/E^2$. So the cross-section will be dominated by the following term

$$\frac{p_1\cdot k_1}{p_1\cdot k_2} = \frac{E+\omega}{E+\omega\cos\theta}. \tag{17}$$

The differential cross-section becomes:

$$d\sigma = \frac{1}{2}\frac{1}{2}\frac{1}{2E}\frac{1}{2\omega}\frac{\omega}{8\pi}\frac{1}{E+\omega}2h^4\frac{E+\omega}{E+\omega\cos\theta}d\cos\theta, \tag{18}$$

which is easily integrable to

$$\sigma = \frac{h^4}{16s}log\left(\frac{s}{M^2}\right), \tag{19}$$

where $s \equiv (E+\omega)^2$.