Stirling approximation for factorial

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Stirling's approximation for the factorial function.

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Consider the following integral:

$$\int_0^\infty dx x^n e^{-x} = \left[(-1)^n \frac{d^n}{d\alpha^n} \int_0^\infty dx e^{-\alpha x}\right]_{\alpha=1} = \left[(-1)^n \frac{d^n}{d\alpha^n} \frac{1}{\alpha}\right]_{\alpha=1} = n!. \tag{1}$$

Taking this definition, we can do the following:

$$n! = \int_0^\infty dx x^n e^{-x} = \int_0^\infty dx e^{n \ln(x) - x}. \tag{2}$$

Let's take a close look at the function in the exponent:

$$u(x) \quad = \quad n \ln(x) - x, \tag{3}$$

as shown in **?@fig-fplot**. This function has its peak value at $x = n$. Note that this function appears in the exponent, under the integral. The dominant contribution to the integral will come from the domain around $x = n$. We can expand $u(x)$ around $x = n$:

$$\begin{aligned} u(x) \quad &= \quad n \ln(x) - x = n \ln(x - n + n) - x = n \ln(n[1 + \frac{x-n}{n}]) - x \\ &\simeq \quad n \left(\ln(n) + \frac{x-n}{n} - \frac{1}{2} \left[\frac{x-n}{n} \right]^2 \right) - x = n \ln(n) - n - \frac{1}{2} \frac{(x-n)^2}{n} \equiv \tilde{u}(x). \end{aligned} \tag{4}$$

The original function and the approximated functions are plotted in **?@fig-fplot**.

From **?@fig-fplot**, we also notice that if we extend the x range to include negative values, the integral would not change much since $e^{\frac{1}{2}\frac{(x-n)^2}{n}}$ is rapidly decaying. Therefore we can change the lower limit of the integral from 0 to $-\infty$ to get:

$$
\begin{aligned}
n! &= \int_0^\infty dx x^n e^{-x} = \int_0^\infty dx e^{u(x)} \simeq \int_0^\infty dx e^{\tilde{u}(x)} = n^n e^{-n} \int_0^\infty dx e^{-\frac{1}{2}\frac{(x-n)^2}{n}} \\
&\simeq n^n e^{-n} \int_{-\infty}^\infty dx e^{-\frac{1}{2}\frac{(x-n)^2}{n}} = n^n e^{-n}\sqrt{2\pi n} = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n . \qquad (5)
\end{aligned}
$$

?@fig-factplot shows the comparison of $n!$ with the Stirling's approximation given in Eq. 5.