Self-Inductance of a Wire

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This article explores the calculation of self-inductance in a wire segment using the Biot-Savart law and energy methods. We derive expressions for both the external and internal contributions to the self-inductance. The external component is calculated by integrating the magnetic flux over a finite region, addressing the inherent challenges of infinite wire assumptions. The internal contribution is determined through energy considerations of the magnetic field within the wire.

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Magnetic Field of a Wire Segment

Figure 1 shows a wire segment of length L carrying a current I and located at the origin.

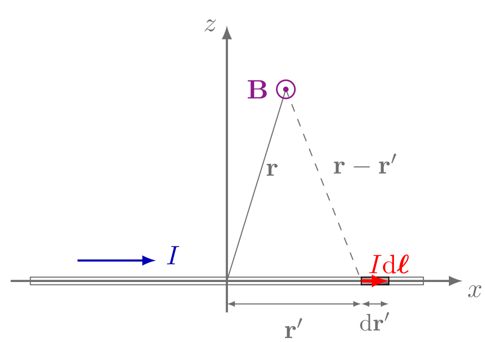

Figure 1: A segment of a wire carrying current I.

The magnetic field at an arbitrary point $\mathbf{r}$ created by a current distribution $\mathbf{J}(\mathbf{r}')$ is given by the Biot-Savart law[1]:

$$\mathbf{B}(\mathbf{r}) \quad = \quad \frac{\mu_0}{4\pi}\int d^3\mathbf{r}'\,\frac{\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{(\mathbf{r}-\mathbf{r}')^3}, \tag{1}$$

$$\mathbf{J}(\mathbf{r}') \quad = \quad \frac{I}{\pi\rho_0^2}\Theta(\rho'-\rho_0)\hat{\mathbf{x}}, \tag{2}$$

Due to the rotational symmetry of the set up, we can compute the magnetic field at $y = 0$ and then rotate it to the desired angle.

$$\mathbf{r}-\mathbf{r}' \quad = \quad z\,\hat{\mathbf{z}} + (x - x')\,\hat{\mathbf{x}}, \tag{3}$$

Let us first consider the magnetic field outside the wire., i.e., $z > \rho_0$.

$$\begin{aligned}
\mathbf{B}(x,z) \quad &= \quad \frac{\mu_0}{4\pi}I\int_{-\frac{L}{2}}^{\frac{L}{2}} dx'\,\frac{\hat{\mathbf{x}}\times(z\,\hat{\mathbf{z}}+(x-x')\,\hat{\mathbf{x}})}{(z^2+(x-x')^2)^{3/2}} = z\,\hat{\mathbf{y}}\frac{\mu_0}{4\pi}I\int_{-\frac{L}{2}}^{\frac{L}{2}} dx'\,\frac{1}{(z^2+(x-x')^2)^{3/2}} \\
&= \quad \hat{\mathbf{y}}\frac{\mu_0}{4\pi z}I\int_{-\frac{L/2-x}{z}}^{\frac{L/2+x}{z}} du\frac{1}{(1+u^2)^{3/2}} = \hat{\mathbf{y}}\frac{\mu_0}{4\pi z}I\left[\frac{u}{\sqrt{1+u^2}}\right]_{-\frac{L/2-x}{z}}^{\frac{L/2+x}{z}} \\
&= \quad \hat{\mathbf{y}}\frac{\mu_0}{4\pi z}I\left[\frac{x+L/2}{\sqrt{z^2+(x+L/2)^2}} - \frac{x-L/2}{\sqrt{z^2+(x-L/2)^2}}\right].
\end{aligned} \tag{4}$$

The Flux Linkage

Now we have to deal with a bit of ambiguity. We will want to integrate over x to compute the linked-flux, however, we can't really integrate from $-\infty$ to ∞ because this will enclose other wires supplying current to the wire we are considering. The exact definition of the linked-flux is heavily dependent on how the circuit is closed. We will assume the boundaries $[-L/2, L/2]$ and integrate over this finite interval. This methodology is consistent with the other method of computing the self-inductance of a wire segment as proposed by Neumann. [2] [3] [4]

The differential flux going through a strip of width dx at z is given by

$$d\Phi(z) \quad = \quad dz\int_{-L/2}^{L/2} dx\mathbf{B}(x,z) = dz\frac{\mu_0 I}{4\pi z}\left[\sqrt{z^2+(x+L/2)^2} - \sqrt{z^2+(x-L/2)^2}\right]_{-L/2}^{L/2} = dz\frac{\mu_0 I}{2\pi z}\left[\sqrt{z^2+L^2} - z\right.$$

Finally, we integrate over z to get the total flux:

$$\Phi \quad = \quad \int_{\rho_0}^{\infty} d\Phi(z) = \frac{\mu_0 I}{2\pi}\int_{\rho_0}^{\infty}\frac{dz}{z}\left[\sqrt{z^2+L^2}-z\right] = \frac{\mu_0 I}{2\pi}\int_{\rho_0}^{\infty} dz\left[\frac{\sqrt{z^2+L^2}}{z} - 1\right] \tag{6}$$

Let's do the harder integral first with the substitution $z = L\sinh t$:

$$\begin{aligned} I &= \int dz \frac{\sqrt{z^2+L^2}}{z} = L\int dt \frac{\cosh^2 t}{\sinh t} = L\int dt \left[\frac{1}{\sinh t} + \sinh t\right] = L\int dt \frac{2}{e^t - e^{-t}} + L\cosh t \\ &= L\int d(e^t)\frac{2}{e^{2t}-1} + L\cosh t = L\left[\ln\left(\frac{e^t+1}{e^t-1}\right) + \sinh t\right] = L\left[\ln\left(\tanh\frac{t}{2}\right) + \cosh t\right] \quad (7) \end{aligned}$$

Now use $\tanh\frac{t}{2} = \frac{\sinh t}{1+\cosh t}$ to get:

$$I = L\ln\left(\frac{z}{L+\sqrt{z^2+L^2}}\right) + \sqrt{z^2+L^2}. \quad (8)$$

Putting it all together we get:

$$\begin{aligned} \Phi &= \frac{\mu_0 I}{2\pi}\left[L\ln\left(\frac{z}{L+\sqrt{z^2+L^2}}\right) + \sqrt{z^2+L^2} - z\right]_{\rho_0}^{\infty} \\ &= \frac{\mu_0 I}{2\pi}\left[-L\ln\left(\frac{\rho_0}{L+\sqrt{\rho_0^2+L^2}}\right) - \sqrt{\rho_0^2+L^2} + \rho_0\right] \simeq \frac{\mu_0 I L}{2\pi}\left[\ln\left(\frac{2L}{\rho_0}\right) - 1\right], \quad (9) \end{aligned}$$

where we assumed $\rho_0 \ll L$. Using the relation $\mathcal{L} = I\Phi$ we get:

$$\mathcal{L} = \frac{\mu_0 L}{2\pi}\left[\ln\left(\frac{2L}{\rho_0}\right) - 1\right]. \quad (10)$$

The Energy in the wire

If the current is distributed uniformly over the wire, the magnetic field inside the wire is given by

$$B(r) = \frac{\mu I}{2\pi}\frac{r}{\rho_0^2}, \quad (11)$$

where μ is the permeability of the wire. The corresponding energy is given by The corresponding energy is given by

$$U = \int d^3\mathbf{r}\frac{B^2}{2\mu} = 2\pi\int_0^{\rho_0} drr\frac{B^2}{2\mu} = \frac{\mu I^2}{4\pi}\int_0^{\rho_0} dr\frac{r^3}{\rho_0^4} = \frac{1}{2}\left[\frac{\mu}{8\pi}\right]I^2 \equiv \frac{1}{2}\mathcal{L}I^2, \quad (12)$$

which results in

$$\mathcal{L} = \frac{\mu}{8\pi}. \quad (13)$$

Putting it all together we get:

$$\mathcal{L} = \frac{\mu_0 L}{2\pi}\left[\ln\left(\frac{2L}{\rho_0}\right) - 1 + \frac{\mu}{4\mu_0}\right]. \quad (14)$$

[1] D. J. Griffiths, *Introduction to electrodynamics*. Pearson, 2013.

[2] F. E. Neumann, "Allgemeine gesetze der inducirten elektrischen ströme," *Abhandlungen der Königlichen Akademie der Wissenschaften zu Berlin*, pp. 1–87, 1847.

[3] E. B. Rosa, "The self and mutual inductances of linear conductors," *Bulletin of the Bureau of Standards*, vol. 4, no. 2, pp. 301–344, 1907.

[4] R. Dengler, "Self inductance of a wire loop as a curve integral." 2013 [Online]. Available: https://arxiv.org/abs/1204.1486